

Studies on Boundary Conditions and Noncommutativity in String Theory

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CERTIFICATE FROM THE SUPERVISOR

This is to certify that the thesis entitled “**Studies on Boundary Conditions and Non-commutativity in String Theory**” submitted by **Sri Arindam Ghosh Hazra** who got his name registered on **20.03.2006** for the award of Ph.D.(Science) degree of Jadavpur University, absolutely based upon his own work under the supervision of **Dr. Biswajit Chakraborty** and that neither this thesis nor any part of it has been submitted for any degree/diploma or any other academic award anywhere before.

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Dedicated
to
My Parents
and
Grand Parents

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List of publications

1. Dual families of noncommutative quantum systems
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2. Non(anti) commutativity for open superstrings
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3. Twisted Galilean symmetry and the Pauli principle at low energies
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4. Normal ordering and noncommutativity in open bosonic strings
B. Chakraborty, S. Gangopadhyay, A. Ghosh Hazra
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5. Noncommutativity in interpolating string: A study of gauge symmetries in a noncommutative framework
S. Gangopadhyay, A. Ghosh Hazra, A. Saha
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6. Normal ordering and non(anti)commutativity in open super strings
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Chapter 1

Introduction and Overview

Noncommutative theories has a long history in physics [1] and has kindled a lot of interest in the past few years owing to the inspiration from string theory [2, 3]. Recent progress in string theory [4, 5, 6] indicates scenarios where our four dimensional space-time with standard model fields corresponds to a D3-brane embedded in a larger manifold. Now, since D-branes correspond, in type II string theories, to the space where the open string endpoints are attached, our space-time would be affected by string boundary conditions. One important consequence is the possible noncommutativity of space-time coordinates at very small length scales since commuting coordinates are incompatible with open string boundary conditions in the presence of anti-symmetric tensor backgrounds [7, 8, 9, 10, 11]. This is one of the main reasons of increasing interest in several aspects of noncommutative quantum mechanics and quantum field theories [2],[12]-[21]. Furthermore, this illustrates the fact that the string boundary conditions may play a crucial role in the phenomenology of four-dimensional physics.

Since the discovery of the role of branes in string theory [22] they have frequently shown unexpected properties. They were first identified as the carriers of R-R charges and very soon after, it was realised that when N of them merge the space-time coordinates normal to them become noncommutative [23] and the $U(N)$ super Yang-Mills theory emerges. Another type of noncommutativity appears in the bound states of branes with fundamental strings and with other branes. It has been shown that such brane bound states correspond to branes with

non-zero background internal gauge fields [23, 24, 25, 26, 27, 28, 29]. The noncommutativity arising in the internal structure of these brane bound states is a consequence of the properties of open strings ending on them. Such open strings satisfy boundary conditions which are neither Neumann nor Dirichlet, but a combination of the two, sometimes referred to as mixed boundary condition [29, 30, 31, 32]. The mixed boundary condition makes the canonical quantisation of the theory non trivial. Imposing the standard commutation relation leads to inconsistency. It has been proposed to remove the inconsistency by relaxing the commutativity of the space coordinates of the open strings along the tangential direction of the brane described by mixed boundary conditions [7, 31, 32, 33]. The procedure of relaxing the commutativity of space coordinates adopted in [30, 33], was to keep the standard algebra of the Fourier modes in the mode expansion.

The noncommutativity observed in the above brane system seemed very similar to that observed by Connes, Douglas and Schwarz [34] in the problem of Matrix Model with non-trivial background three form. They studied compactification of Matrix theory on a noncommutative torus and realised that it corresponds to the Matrix theory in such backgrounds. Motivated by this observation, Ardlan *et al.* showed that [31, 33], the noncommutativity can be derived within the string theory by wrapping branes with non zero $B_{\mu\nu}$ background field on the compactification torus.

Different approaches have been adopted to obtain this noncommutativity. A Hamiltonian operator treatment was provided in [7] and a world sheet approach in [35]. Also, an alternative Hamiltonian (Dirac [36]) approach based on regarding the Boundary Conditions as constraints was given in [37, 38]; the corresponding Lagrangian (symplectic) version being done in [39]. The interpretation of the boundary condition as primary constraints usually led to an infinite tower of second class constraints [40], in contrast to the usual Dirac formulation of constrained systems [36, 41]. Some other approaches to this problem have been discussed in [42, 43, 44, 45, 46, 47, 48, 49].

On the other hand, it has also been shown that non-commutativity can be obtained in a more transparent manner by modifying the canonical Poisson bracket structure, so that it is

compatible with the boundary condition [10, 50]. In this approach, the boundary conditions are not treated as constraints. This is similar in spirit to the treatment of Hanson, Regge and Teitelboim [41], where modified Poisson brackets were obtained for the free Nambu-Goto string, in the orthonormal gauge, which is the counterpart of the conformal gauge in the free Polyakov string. Those studies were, however, restricted to the case of the bosonic string and membrane only. Proceeding further Jing and Long [51] obtained the Poisson brackets among the Fourier components using the Faddeev-Jackiw symplectic formalism [52], so that they are compatible with these boundary conditions. Using this they obtained the Poisson brackets among the open string coordinates revealing the noncommutative structure in the string end points.

1.1 Structure of the thesis

The central theme of this thesis is noncommutativity in string theory. We explore in detail how noncommutative structures can emerge in case of the interacting bosonic string and even in the fermionic sector of superstring theory. We have shown in various approaches that string coordinates must be noncommutative in order to be compatible with boundary conditions. These noncommutative structures lead to new involutive algebra of constraints but generate same Virasoro algebra, indicating the internal consistency of our analysis.

On the other hand the action for a string can be chosen, in analogy with the relativistic particle as the proper area of the world sheet swept out by the dynamical string. This gives the Nambu-Goto formalism which, however, poses problems in quantisation. A redundant description, where the world sheet metric coefficients are considered as independent fields, has been shown by Polyakov [53] to be particularly suitable in this context. The ensuing action is known as the Polyakov action. The equivalence between the two approaches can be established on shell by solving the independent metric in the Polyakov action. The classical correspondence is assumed to lead to equivalent result at the quantum level. Understanding this correspondence from different view points will, naturally, be useful.

- We start with the gauge independent analysis of Polyakov string and give a review, based

on [10], of the emergence of noncommutativity in the context of an open string. Here the authors, do not treat the boundary conditions as constraints, but show that they can be systematically implemented by modifying the canonical Poisson bracket structure. We follow the same methodology to obtain the noncommutativity among the string coordinates for both interpolating and super strings in the following chapters. This is the subject matter of chapter 2.

A deeper connection between Polyakov action and Nambu-Goto string action has been demonstrated in [10] by constructing a Lagrangian description which interpolates between the Nambu-Goto and Polyakov forms in the free case. The interpolating theory thus offers a unified pictures for understanding different features of the basic structures including their various symmetry properties. In this sense, therefore, it is more general than either the Nambu-Goto or Polyakov formulation. An added advantage is that it illuminates the passage from the Nambu-Goto form to the Polyakov form, which is otherwise lacking. In this context it may be noted that the Polyakov action has the additional Weyl invariance which Nambu-Goto action does not have. The interpolating action, which does not presuppose Weyl invariance, thus offers a proper platform of discussing the equivalence of the two actions. It also explains the emergence of the Weyl invariance in a natural way. In this work we study the interpolating formalism both in free and interacting case.

- In chapter 3 we derive a master action for interacting bosonic strings, interpolating between the Nambu-Goto and Polyakov formalism. Modification of the basic poisson bracket structure compatible with boundary conditions followed by the emergence of the noncommutativity is shown in this formalism (in case of both free and interacting strings) following the approach discussed in chapter 2 and [10, 41, 50]. Our results go over smoothly to the Polyakov version once the proper identifications are made. This noncommutativity leads to a new involutive constraint algebra which is markedly different from that obtained in second chapter [10]. With the above results at our disposal, we go over to the study of gauge symmetry in the noncommutative framework. Owing to the new constraint algebra we find surprising changes in the structure constants of the theory. Finally, we compute the gauge variations of the fields and show the

underlying unity of diffeomorphism with the gauge symmetry in the noncommutative framework [54].

So far our attention was basically confined to the classical level. We now extend parts of the foregoing analysis to the quantum level. To that end, recall that in quantum field theory, products of quantum fields at the same space-time points are in general singular objects. The same thing happens in string theory if one multiplies position operators, that can be taken as conformal fields on the world sheet. This situation is well known and one can remove the singular part of the operator products by defining normal ordered well behaved objects [53]. Normal ordered products of operators are usually defined so as to satisfy the classical equations of motion at quantum level.

Recently Braga *et al.* [55] defined normal ordered products for open string position operators that additionally satisfy the boundary conditions. This way one can define a normal ordering that will be valid also at string end-points.

- In the 4th chapter, noncommutativity in an open bosonic string moving in the presence of a background Neveu-Schwarz two-form field $B_{\mu\nu}$ is investigated in a conformal field theory approach. The mode algebra is first obtained using the newly proposed normal ordering, which satisfies both equations of motion and boundary conditions. Using these the commutator among the string coordinates is obtained. Interestingly, this new normal ordering yields the same algebra between the modes as the one satisfying only the equations of motion. In this approach, we find that noncommutativity originates more transparently and our results match with the existing results in the literature [56].

Compared to the bosonic string theory, the supersymmetric case as well as superstring theory has received less attention. In Ref [8], the authors discussed the fermionic part without resorting to the dynamical properties. They start from the bosonic results by supersymmetric transformations. In [57], the authors find that in order to keep the supersymmetry unbroken in an open string's end points, it is necessary to add a proper boundary term to the supersymmetric action. Authors of [58] work in the discrete version. They take the boundary conditions as constraints and then Fadeev Jackiw method is employed to get the anti-poisson brackets among

the string coordinates. Jing also obtained the anti-poisson brackets in [59] by following the same methodology of their bosonic paper [51].

- In chapter 5 we start with the Ramond Neveu Schwarz superstring action in the conformal gauge and discuss the super constraint structure of the theory. The non(anti)commutativity of the theory is then revealed in the conventional Hamiltonian framework by following [10, 41, 50]. We have obtained this expressions of non(anti)commutative structure for an open superstring by modifying the canonical bracket, so that it is compatible with the boundary conditions. We find that the non(anti)commutative structures not only appear for an open string moving in the antisymmetric background field but also in the free case. This is indeed a new result. We have also shown that this symplectic structure leads to a new involutive structure for the super constraint algebra at the classical level [60].

- In Chapter 6 we extend our methodology discussed in chapter 4 to analyse an open super string propagating freely and one moving in a constant antisymmetric background field [61]. We start by reviewing the recent results involving new normal ordered products of fermionic operators [62]. The mode algebra is then obtained using the newly proposed normal ordering, which satisfies both equations of motion and boundary conditions. Finally we obtain same anti-commutators among the string coordinates by using the mode algebra.

- It is surprising that all the studies in the context of superstring theory is based on Ramond boundary conditions. But as is well known in the context of fermionic string there is a choice between Ramond boundary conditions and Neveu Schwarz boundary conditions. In chapter 5 and chapter 6 we have a detailed discussion on the problem of non(anti)commutativity on the basis of Ramond boundary conditions. In chapter 7, we extend our methodology to the superstring satisfying the Neveu Schwarz boundary conditions [63].

Finally, in Chapter 8 we summarise the important results.

This thesis is based on the following publications.

1. Non(anti) commutativity for open superstrings [60]
B. Chakraborty, S. Gangopadhyay, **A. Ghosh Hazra**, F. G. Scholtz
Phys. Lett. B 625 (2005) 302-312.
2. Normal ordering and noncommutativity in open bosonic strings [56]
B. Chakraborty, S. Gangopadhyay, **A. Ghosh Hazra**
Phys. Rev. D **74** (2006) 105011.
3. Noncommutativity in interpolating string: A study of gauge symmetries in a noncommutative framework [54]
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4. Normal ordering and non(anti)commutativity in open super strings [61]
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Phys. Rev. D **75** (2007) 065026.
5. String non(anti)commutativity for Neveu-Schwarz boundary conditions [63]
C. Chatterjee, S. Gangopadhyay, **A. Ghosh Hazra**, S. Samanta
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Chapter 2

Review of Bosonic String

An intriguing connection between string theory, noncommutative geometry and noncommutative Yang-Mills theory was revealed in [2]. The study of open string, in the presence of a background Neveu-Schwarz two-form field $B_{\mu\nu}$, leads to a noncommutative(NC) structure which manifests in the noncommutativity at the end points of the string which are attached to D-branes. Different approaches have been adopted to obtain this result.

In this present chapter, we discuss the Polyakov action and also the essential result of [10] in which the authors provide an exhaustive analysis of the noncommutativity in open string theory moving in the presence of a constant Neveu-Schwarz field, in the conventional Hamiltonian framework. In contrast to the usual studies, this model of string theory is very general in the sense that no gauge is fixed at the beginning. Let us recall that all computations of noncommutativity, mentioned before, were done in the conformal gauge. This gauge independent analysis yields a new noncommutative structure, which correctly reduces to the usual one in conformal gauge. This shows the compatibility of the present analysis with the existing literature. In the general case, the noncommutativity is manifested at all points of the string, in contrast to conformal gauge results where it appears only at the boundaries. Indeed, in this gauge independent scheme, one finds a noncommutative algebra among the coordinates, even for a free string, a fact that was not observed before. Expectedly, this noncommutativity vanishes in the conformal gauge. Note however, that there is no gauge for which noncommutativity vanishes

in the interacting theory.

At the outset, let us point out the crucial difference between existing Hamiltonian analysis [9] and this approach. This is precisely in the interpretation of the boundary conditions(BC) arising in the string theory. The general consensus has been to consider the boundary conditions as primary constraints of the theory and attempt a conventional Dirac constraint analysis [36]. The aim is to induce the noncommutativity in the form of Dirac Brackets between coordinates. The subsequent analysis turns out to be ambiguous since it involves the presence of $\delta(0)$ -like factors, (see Chu and Ho in [9]). Different results are obtained depending on the interpretation of these factors.

Here on the other hand we do not treat the BCs as constraints, but show that they can be systematically implemented by modifying the canonical Poisson Bracket(PB) structure. In this sense this approach is quite similar in spirit to that of Hanson, Regge and Teitelboim [41], where modified PBs were obtained for the free Nambu-Goto string, in the orthonormal gauge, which is the counterpart of the conformal gauge in the free Polyakov string.

2.1 The free string in Polyakov formalism

In this section, we analyze the Polyakov formulation of the free string. The Polyakov action for a free bosonic string reads,

$$S_P = -\frac{1}{2} \int_{-\infty}^{+\infty} d\tau \int_0^\pi d\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X_\mu \quad (2.1)$$

where τ and σ are the usual world-sheet parameters and g_{ab} , up to a Weyl factor, is the induced metric on the world-sheet. $X^\mu(\sigma)$ are the string coordinates in the D-dimensional Minkowskian target space with metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, \dots, 1)$.

This action has the following symmetries:

- 1. D-dimensional Poincaré invariance:

$$\begin{aligned} X'^\mu(\tau, \sigma) &= \Lambda^\mu{}_\nu X^\nu(\tau, \sigma) + a^\mu \\ g'_{ab}(\tau, \sigma) &= g_{ab}(\tau, \sigma) \end{aligned} \quad (2.2)$$

- 2. Diffeomorphism Invariance:

$$\begin{aligned} X'^{\mu}(\tau', \sigma') &= X^{\mu}(\tau, \sigma) \\ \frac{\partial \sigma'^c}{\partial \sigma^a} \frac{\partial \sigma'^d}{\partial \sigma^b} g'_{cd}(\tau', \sigma') &= g_{ab}(\tau, \sigma) \end{aligned} \quad (2.3)$$

for new coordinates $\sigma'^a(\tau, \sigma)$.

- 3. Two-dimensional Weyl invariance:

$$\begin{aligned} X'^{\mu}(\tau, \sigma) &= X^{\mu}(\tau, \sigma) \\ g'_{ab}(\tau, \sigma) &= \exp(2\omega(\tau, \sigma)) g_{ab}(\tau, \sigma) \end{aligned} \quad (2.4)$$

for arbitrary $\omega(\tau, \sigma)$.

Here we carry out our analysis in the complete space by regarding both X^{μ} and g_{ab} as independent dynamical variables [64]. The canonical momenta are,

$$\begin{aligned} \Pi_{\mu} &= \frac{\delta \mathcal{L}_P}{\delta (\partial_{\tau} X^{\mu})} = -\sqrt{-g} \partial_{\tau} X_{\mu} \\ \pi_{ab} &= \frac{\delta \mathcal{L}_P}{\delta (\partial_{\tau} g^{ab})} = 0. \end{aligned} \quad (2.5)$$

It is clear that while Π_{μ} is a genuine momenta, $\pi_{ab} \approx 0$ are the primary constraints of the theory. The conservation of the above primary constraints leads to the secondary constraints $\Omega_1(\sigma)$ and $\Omega_2(\sigma)$. These secondary constraints also follow from the equation obtained by varying g_{ab} since this is basically a Lagrange multiplier. This imposes the vanishing of the symmetric energy-momentum tensor,

$$T_{ab} = \frac{2}{\sqrt{-g}} \frac{\delta S_P}{\delta g^{ab}} = -\partial_a X^{\mu} \partial_b X_{\mu} + \frac{1}{2} g_{ab} g^{cd} \partial_c X^{\mu} \partial_d X_{\mu} = 0. \quad (2.6)$$

Because of the Weyl invariance, the energy-momentum tensor is traceless,

$$T^a_a = g^{ab} T_{ab} = 0$$

so that only two components of T_{ab} are independent. These components, which are the constraints of the theory, are given by,

$$\begin{aligned}\Omega_1(\sigma) &= g T^{00} = -T_{11} = \left(\Pi^2(\sigma) + X'^2(\sigma) \right) = 0 \\ \Omega_2(\sigma) &= \sqrt{-g} T^0_1 = \Pi(\sigma) \cdot X'(\sigma) = 0\end{aligned}\tag{2.7}$$

The canonical Hamiltonian obtained from (2.1) by a Legendre transformation is given by,

$$H = \int d\sigma \sqrt{-g} T^0_0 = \int d\sigma \sqrt{-g} \left(\frac{1}{2g_{11}} \Omega_1(\sigma) + \frac{g_{01}}{\sqrt{-g} g_{11}} \Omega_2(\sigma) \right)\tag{2.8}$$

expectedly, the Hamiltonian turns out to be a linear combination of the constraints.

Just as variation of g_{ab} yields the constraints, variation of X^μ gives the equation of motion,

$$\partial_a (\sqrt{-g} g^{ab} \partial_b X^\mu) = 0\tag{2.9}$$

Finally, there is a mixed BC,

$$\partial^\sigma X^\mu(\tau, \sigma)|_{\sigma=0, \pi} = 0\tag{2.10}$$

In the covariant form involving phase space variables, this is given by

$$\left(\partial_\sigma X^\mu + \sqrt{-g} g^{01} \Pi^\mu \right) |_{\sigma=0, \pi} = 0.\tag{2.11}$$

The non trivial basic PBs of the theory are:

$$\begin{aligned}\{X^\mu(\tau, \sigma), \Pi_\nu(\tau, \sigma')\} &= \delta^\mu_\nu \delta(\sigma - \sigma') \\ \{g_{ab}(\tau, \sigma), \pi^{cd}(\tau, \sigma')\} &= \frac{1}{2} (\delta^c_a \delta^d_b + \delta^d_a \delta^c_b) \delta(\sigma - \sigma')\end{aligned}\tag{2.12}$$

where $\delta(\sigma - \sigma')$ is the usual one-dimensional Dirac delta function. From the basic PB, it is easy to generate the following first class (involutive) algebra,

$$\begin{aligned}\{\Omega_1(\sigma), \Omega_1(\sigma')\} &= 4 (\Omega_2(\sigma) + \Omega_2(\sigma')) \partial_\sigma \delta(\sigma - \sigma'), \\ \{\Omega_2(\sigma), \Omega_1(\sigma')\} &= (\Omega_1(\sigma) + \Omega_1(\sigma')) \partial_\sigma \delta(\sigma - \sigma'), \\ \{\Omega_2(\sigma), \Omega_2(\sigma')\} &= (\Omega_2(\sigma) + \Omega_2(\sigma')) \partial_\sigma \delta(\sigma - \sigma').\end{aligned}\tag{2.13}$$

2.2 Modified brackets for the Polyakov string

Let us again consider the BCs for the Polyakov string,

$$\left(\partial_\sigma X^\mu + \sqrt{-g}g^{01}\Pi^\mu\right)|_{\sigma=0,\pi} = 0. \quad (2.14)$$

It is easily seen that the above BCs are incompatible with the first of the basic PBs (2.12). Hence the brackets should be modified suitably. The modification of PBs can be done in spirit to the treatment of Hanson *et al.* [41], where modified PBs were obtained for the free Nambu-Goto string.

We would also like to mention that there is an apparent contradiction of the constraint $\pi_{ab} \approx 0$ with the second PB (2.12). However this equality is valid in Dirac's "weak" sense only, so that it can be set equal to zero only after the relevant brackets have been computed. These weak equalities will be designated by \approx , rather than an equality, which is reserved only for a strong equality. In this sense, therefore, there is no clash between this constraint and the relevant PB. Indeed, we can even ignore the canonical pair (g_{ab}, π^{cd}) from the basic PB.

The situation is quite similar to usual electrodynamics. There the Lagrange multiplier is A_0 , which corresponds to g_{ab} in the string theory. The multiplier A_0 enforces the Gauss constraint just as g_{ab} enforces the constraints Ω_1 and Ω_2 . Furthermore, the Gauss constraint generates the time independent gauge transformations, while Ω_1, Ω_2 generate the diffeomorphism transformations.

The BC (2.14), on the other hand, is not a constraint in the Dirac sense [36], since it is applicable only at the boundary. Thus, there has to be an appropriate modification in the PB, to incorporate this condition. This is not unexpected and occurs, for instance, in the example of a free scalar field $\phi(x)$ in $(1+1)$ dimension, subjected to periodic BC of period, say, 2π ($\phi(t, x+2\pi) = \phi(t, x)$). There the PB between the field $\phi(t, x)$ and its conjugate momentum $\pi(t, x)$ are given by,

$$\{\phi(t, x), \pi(t, y)\} = \delta_P(x - y) \quad (2.15)$$

where,

$$\delta_P(x - y) = \delta_P(x - y + 2\pi) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in(x-y)} \quad (2.16)$$

is the periodic delta function of period 2π [65] and occurs in the closure properties of the basis functions e^{inx} for the space of square integrable functions, defined on the unit circle S^1 . This periodic delta function is related to the usual Dirac delta function as $\delta_P(x - y) = \sum_{n \in \mathbb{Z}} \delta(x - y + 2\pi n)$

Before discussing the mixed type condition (2.14), that emerged in a completely gauge independent formulation of the Polyakov action, consider the simpler Neumann type condition $(\partial_\sigma X^\mu)|_{\sigma=0,\pi} = 0$ in an orthonormal (conformal) gauge. It is easy to find the solutions to the equations of motion (2.9) which are compatible with the Neumann BCs

$$X^\mu(\tau, \sigma) = x^\mu + p^\mu \tau + i \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau} \cos(n\sigma) \quad (2.17)$$

Reality of $X^\mu(\tau, \sigma)$ implies that x, p are real and

$$\alpha_n^{\mu*} = \alpha_{-n}^\mu \quad \text{for } n \neq 0. \quad (2.18)$$

We enlarge the domain of definition of the bosonic field X^μ from $[0, \pi]$ to $[-\pi, \pi]$ by observing the fact

$$X^\mu(\tau, -\sigma) = X^\mu(\tau, \sigma) \text{ under } \sigma \rightarrow -\sigma \quad (2.19)$$

which further yields

$$X^\mu(-\pi) = X^\mu(\pi). \quad (2.20)$$

Now we start by noting that the usual properties of a delta function is also satisfied by $\delta_P(x)$ (2.16),

$$\int_{-\pi}^{\pi} dx' \delta_P(x' - x) f(x') = f(x) \quad (2.21)$$

for any periodic function $f(x) = f(x + 2\pi)$ defined in the interval $[-\pi, \pi]$. Then by using (2.19), the above integral (2.21) reduces to the following:

$$\int_0^\pi d\sigma' \Delta_+(\sigma', \sigma) X^\mu(\sigma') = X^\mu(\sigma) \quad (2.22)$$

where

$$\Delta_+(\sigma', \sigma) = \delta_P(\sigma' - \sigma) + \delta_P(\sigma' + \sigma). \quad (2.23)$$

Using (2.16), the explicit form of $\Delta_+(\sigma', \sigma)$ can be given as,

$$\Delta_+(\sigma', \sigma) = \frac{1}{\pi} + \frac{1}{\pi} \sum_{n \neq 0} \cos(n\sigma') \cos(n\sigma). \quad (2.24)$$

It thus follows that the appropriate PB is given by,

$$\{X^\mu(\tau, \sigma), \Pi_\nu(\tau, \sigma')\} = \delta_\nu^\mu \Delta_+(\sigma', \sigma). \quad (2.25)$$

It is clearly consistent with Neumann BC as $\partial_\sigma \Delta_+(\sigma, \sigma')|_{\sigma=0, \pi} = \partial_{\sigma'} \Delta_+(\sigma, \sigma')|_{\sigma=0, \pi} = 0$ and is automatically satisfied. Observe also that the other brackets

$$\{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} = 0 \quad (2.26)$$

$$\{\Pi^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')\} = 0 \quad (2.27)$$

are already consistent with the Neumann BCs and hence remain unchanged.

For a gauge independent analysis, we take recourse to the mixed condition (2.14). A simple inspection shows that this is also compatible with the modified brackets (2.25 , 2.27), but not with (2.26). Hence the bracket among the coordinates should be altered suitably. We therefore make an ansatz,

$$\{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} = C^{\mu\nu}(\sigma, \sigma') \quad (2.28)$$

where,

$$C^{\mu\nu}(\sigma, \sigma') = -C^{\nu\mu}(\sigma', \sigma).$$

Imposing the BC (2.14) on this algebra, we get,

$$\begin{aligned} \partial_{\sigma'} C^{\mu\nu}(\sigma, \sigma')|_{\sigma'=0, \pi} &= \partial_\sigma C^{\mu\nu}(\sigma, \sigma')|_{\sigma=0, \pi} \\ &= -\sqrt{-g}g^{01}\{\Pi^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} \\ &= \sqrt{-g}g^{01}\eta^{\mu\nu}\Delta_+(\sigma, \sigma') \end{aligned} \quad (2.29)$$

For an arbitrary form of the metric tensor, it might be technically problematic to find a solution for $C^{\mu\nu}(\sigma, \sigma')$. However, for a restricted class of metric¹ that satisfy

$$\partial_\sigma g_{ab} = 0$$

it is possible to give a quick solution of $C^{\mu\nu}(\sigma, \sigma')$ as,

$$C^{\mu\nu}(\sigma, \sigma') = \sqrt{-g} g^{01} \eta^{\mu\nu} [\Theta(\sigma, \sigma') - \Theta(\sigma', \sigma)] \quad (2.30)$$

where the generalised step function $\Theta(\sigma, \sigma')$ satisfies,

$$\partial_\sigma \Theta(\sigma, \sigma') = \Delta_+(\sigma, \sigma') \quad (2.31)$$

An explicit form of Θ is given by [41],

$$\Theta(\sigma, \sigma') = \frac{\sigma}{\pi} + \frac{1}{\pi} \sum_{n \neq 0} \frac{1}{n} \sin(n\sigma) \cos(n\sigma') , \quad (2.32)$$

having the properties,

$$\begin{aligned} \Theta(\sigma, \sigma') &= 1 \quad \text{for } \sigma > \sigma' , \\ \Theta(\sigma, \sigma') &= 0 \quad \text{for } \sigma < \sigma' . \end{aligned} \quad (2.33)$$

Using these relations, the simplified structure of noncommutative algebra follows,

$$\begin{aligned} \{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} &= 0 \quad \text{for } \sigma = \sigma' \\ \{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} &= \pm \sqrt{-g} g^{01} \eta^{\mu\nu} \quad \text{for } \sigma \neq \sigma' \end{aligned} \quad (2.34)$$

respectively. Thus a noncommutative algebra for distinct coordinates $\sigma \neq \sigma'$ of the string emerges automatically in a free string theory if a gauge independent analysis is carried out like this. But this non-commutativity can be made to vanish in gauges like conformal gauge, where $g^{01} = 0$, thereby restoring the usual commutative structure.

Now using the modified basic brackets we obtain the following involutive constraint algebra²

$$\begin{aligned} \{\Omega_1(\sigma), \Omega_1(\sigma')\} &= \Omega_1(\sigma') \partial_\sigma \Delta_+(\sigma, \sigma') + \Omega_1(\sigma) \partial_\sigma \Delta_-(\sigma, \sigma') \\ \{\Omega_1(\sigma), \Omega_2(\sigma')\} &= (\Omega_2(\sigma) + \Omega_2(\sigma')) \partial_\sigma \Delta_+(\sigma, \sigma') \\ \{\Omega_2(\sigma), \Omega_2(\sigma')\} &= 4 (\Omega_1(\sigma) \partial_\sigma \Delta_+(\sigma, \sigma') + \Omega_1(\sigma') \partial_\sigma \Delta_-(\sigma, \sigma')) . \end{aligned} \quad (2.35)$$

¹Such conditions also follow from a standard treatment of the light-cone gauge [53]

²Note that there were some errors in [10]

A crucial intermediate step in the above derivation is to use the relation,

$$\{X'^\mu(\sigma), X'^\nu(\sigma')\} = 0 \quad (2.36)$$

which follows from the basic bracket (2.34) [10].

2.3 Interacting Polyakov string

The Polyakov action for a bosonic string moving in the presence of a constant background Neveu-Schwarz two-form field $B_{\mu\nu}$ is given by,

$$S_P = -\frac{1}{2} \int d\tau d\sigma \left(\sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X_\mu + e \epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \right) \quad (2.37)$$

where $\epsilon^{01} = -\epsilon^{10} = +1$. A usual canonical analysis leads to the following set of primary first class constraints,

$$\begin{aligned} gT^{00} &= \frac{1}{2} \left[(\Pi_\mu + e B_{\mu\nu} \partial_\sigma X^\nu)^2 + (\partial_\sigma X)^2 \right] \approx 0 \\ \sqrt{-g} T^0_1 &= \Pi_\mu \partial_\sigma X \approx 0 \end{aligned} \quad (2.38)$$

where

$$\Pi_\mu = -\sqrt{-g} \partial^\tau X_\mu + e B_{\mu\nu} \partial_\sigma X^\nu \quad (2.39)$$

is the momentum conjugate to X^μ . The boundary condition written in terms of phase-space variables is

$$\left[\partial_\sigma X_\mu + \Pi^\rho (N M^{-1})_{\rho\mu} \right]_{\sigma=0,\pi} = 0 \quad (2.40)$$

where,

$$\begin{aligned} M^\rho_\mu &= \frac{1}{g_{11}} \left[\delta^\rho_\mu - \frac{2e}{\sqrt{-g}} g_{01} B^\rho_\mu + e^2 B^{\rho\nu} B_{\nu\mu} \right] \\ N_{\nu\mu} &= -\frac{g^{01}}{g^{00} \sqrt{-g}} \eta_{\nu\mu} - \frac{1}{g_{11}} e B_{\nu\mu} \end{aligned} \quad (2.41)$$

are two matrices.

The $\{X^\mu, \Pi_\nu\}$ Poisson bracket is the same as that of the free string whereas considering the general structure (2.28) and exploiting the above boundary condition, one obtains

$$\partial_\sigma C_{\mu\nu}(\sigma, \sigma') \big|_{\sigma=0,\pi} = (NM^{-1})_{\nu\mu} \Delta_+(\sigma, \sigma') \big|_{\sigma=0,\pi} . \quad (2.42)$$

As in the free case, we restrict to the class of metrics defined satisfying $\partial_\sigma g_{ab} = 0$, the above equation has a solution

$$\begin{aligned} C_{\mu\nu}(\sigma, \sigma') &= \frac{1}{2}(NM^{-1})_{(\nu\mu)} [\Theta(\sigma, \sigma') - \Theta(\sigma', \sigma)] \\ &\quad + \frac{1}{2}(NM^{-1})_{[\nu\mu]} [\Theta(\sigma, \sigma') + \Theta(\sigma', \sigma) - 1] . \end{aligned} \quad (2.43)$$

where $(NM^{-1})_{(\nu\mu)}$ the symmetric and $(NM^{-1})_{[\nu\mu]}$ the antisymmetric part of $(NM^{-1})_{\nu\mu}$. The modified algebra is gauge dependent; it depends on the choice of the metric. However, there is no choice, for which the non-commutativity vanishes. To show this, note that the origin of the non-commutativity is the presence of non-vanishing $N^{\nu\mu}$ in the BC (2.40). Vanishing $N^{\nu\mu}$ would make $B_{\mu\nu}$ and $\eta_{\mu\nu}$ proportional to each other which obviously cannot happen, as the former is an antisymmetric and the latter is a symmetric tensor. Hence non-commutativity will persist for any choice of world-sheet metric g_{ab} .

2.4 Summary

In this chapter we have discussed the Polyakov string and derived the expressions for a noncommutative algebra [10], that are more general than the standard results found in the conformal gauge. The origin of any modification in the usual Poisson algebra is the presence of boundary conditions. This phenomenon is quite well known for a free scalar field subjected to periodic boundary conditions. We showed that its exact analogue is the conformal gauge fixed free string, where the boundary condition is of Neuman-type. This led to a modification only in the $\{X^\mu(\sigma), \Pi_\nu(\sigma')\}$ algebra, where the usual Dirac delta function got replaced by $\Delta_+(\sigma, \sigma')$. Using certain algebraic consistency requirements, we showed that the boundary conditions in the free theory naturally led to a noncommutative structure among the coordinates. This non-commutativity, however, vanishes in the conformal gauge, as expected. The same technique

was adopted for the interacting string. Here, on contrast, we find that there is a genuine noncommutativity at the string end points and can not be made to vanish in any gauge.

Chapter 3

Interpolating string: A study of gauge symmetries in a noncommutative framework

The dynamics of a bosonic string is described either by Nambu–Goto(NG) or Polyakov action. Both these actions, though very well-known in the literature, poses certain degrees of difficulty. NG formalism is inconvenient for path integral quantisation whereas Polyakov action involves many redundant degrees of freedom. However, yet another formulation, interpolating between these two versions of string action, has also been put forward in the literature [10, 54]. This interpolating Lagrangian, in a certain sense, is a better description of the theory in the sense that it neither objects to quantization nor has as many redundancies as in the Polyakov version. Further, it gives a perfect platform to study the gauge symmetries vis-à-vis reparametrisation symmetries of the various free string actions by a constrained Hamiltonian approach [66, 67].

In the present chapter, acknowledging the above facts, we derive a master action for interacting bosonic strings, interpolating between the NG and Polyakov formalism. Modification of the basic PB structure compatible with BC(s) followed by the emergence of the noncommutativity is shown in this formalism (in case of both free and interacting strings) following the approach discussed in previous chapter [10, 50, 60]. Our results go over smoothly to the Polyakov version

once proper identifications are made. These modified PBs lead to a new involutive constraint algebra which is markedly different from that given in [10]. With the above results at our disposal, we go over to the study of gauge symmetry in the NC framework. Owing to the new constraint algebra we find surprising changes in the structure constants of the theory. Finally, we compute the gauge variations of the fields and show the underlying unity of diffeomorphism with the gauge symmetry in the NC framework.

3.1 The interacting theory: Nambu-Goto Formulation

Although the Polyakov and NG formulations for free strings are regarded to be classically equivalent, there are some subtle issues. Indeed, the structures of BCs in the two formulations are different. Also more complications are expected in the presence of interactions. In this section, we analyse the NG formulation of the interacting bosonic string. As we shall see in the next section, this is essential in the construction of the Interpolating Lagrangian for the interacting string. The NG action for a bosonic string moving in the presence of a constant background Neveu-Schwarz two-form field $B_{\mu\nu}$ reads:

$$S_{NG} = \int_{-\infty}^{\infty} d\tau \int_0^{\pi} d\sigma \left[\mathcal{L}_0 + e B_{\mu\nu} \dot{X}^{\mu} X'^{\nu} \right] \quad (3.1)$$

where \mathcal{L}_0 is the free NG Lagrangian density given by

$$\mathcal{L}_0 = - \left[(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2 \right]^{\frac{1}{2}}. \quad (3.2)$$

The Euler-Lagrange equations and BC obtained by varying the action read:

$$\begin{aligned} \dot{\Pi}^{\mu} + K'^{\mu} &= 0 \\ K^{\mu}|_{\sigma=0,\pi} &= 0 \end{aligned} \quad (3.3)$$

where,

$$\begin{aligned} \Pi_{\mu} &= \frac{\partial \mathcal{L}}{\partial \dot{X}^{\mu}} = \mathcal{L}_0^{-1} \left(-X'^2 \dot{X}_{\mu} + (\dot{X} \cdot X') X'_{\mu} \right) + e B_{\mu\nu} X'^{\nu} \\ K_{\mu} &= \frac{\partial \mathcal{L}}{\partial X'^{\mu}} = \mathcal{L}_0^{-1} \left(-\dot{X}^2 X'_{\mu} + (\dot{X} \cdot X') \dot{X}_{\mu} \right) - e B_{\mu\nu} \dot{X}^{\nu}. \end{aligned} \quad (3.4)$$

Note that Π_μ is the canonically conjugate momentum to X^μ . The first of (2.12) is the only nontrivial PB of NG theory. The primary constraints of the theory are:

$$\begin{aligned}\Omega_1 &= \Pi_\mu X'^\mu = 0 \\ \Omega_2 &= (\Pi_\mu - eB_{\mu\nu}X'^\nu)^2 + X'^2 = 0\end{aligned}\tag{3.5}$$

and they generate the same first class (involutive) algebra as that of Polyakov string (2.13)

Now as happens for a reparametrisation invariant theory, the canonical Hamiltonian density defined by a Legendre transform vanishes

$$\mathcal{H}_c = \Pi_\mu \dot{X}^\mu - \mathcal{L} = 0.\tag{3.6}$$

This can be easily seen by substituting (3.4) in (3.6). The total Hamiltonian density is thus given by a linear combination of the first class constraints (3.5):

$$\mathcal{H}_T = -\rho\Omega_1 - \frac{\lambda}{2}\Omega_2\tag{3.7}$$

where ρ and λ are Lagrange multipliers. It is easy to check that time preserving the primary constraints yields no new secondary constraints. Hence the total set of constraints of the NG theory is given by the first-class system (3.5).

As in the case of Polyakov string, here also we enlarge the domain of definition of the bosonic field X^μ from $[0, \pi]$ to $[-\pi, \pi]$ in order to write down the generators of τ and σ reparametrisation in a compact form. We define

$$X^\mu(\tau, -\sigma) = X^\mu(\tau, \sigma) \quad ; \quad B_{\mu\nu} \rightarrow -B_{\mu\nu} \text{ under } \sigma \rightarrow -\sigma.\tag{3.8}$$

The second condition implies that $B_{\mu\nu}$, albeit a constant, transforms as a pseudo scalar under $\sigma \rightarrow -\sigma$ in the extended interval. This ensures that the interaction term $eB_{\mu\nu}\dot{X}^\mu X'^\nu$ in (3.1) remains invariant under $\sigma \rightarrow -\sigma$ like the free NG Lagrangian density \mathcal{L}_0 (3.2). Consistent with this, we have

$$\Pi^\mu(\tau, -\sigma) = \Pi^\mu(\tau, \sigma), \quad X'^\mu(\tau, -\sigma) = -X'^\mu(\tau, \sigma).\tag{3.9}$$

Now, from (3.5), (3.8) we note that the constraints $\Omega_1(\sigma) = 0$ and $\Omega_2(\sigma) = 0$ are odd and even respectively under $\sigma \rightarrow -\sigma$. Now demanding the total Hamiltonian density \mathcal{H}_T (3.7) also

remains invariant under $\sigma \rightarrow -\sigma$, one finds that ρ and λ must be odd and even respectively under $\sigma \rightarrow -\sigma$.

We may then write the generator of all τ and σ reparametrisation as the functional [41]:

$$L[f] = \frac{1}{2} \int_0^\pi d\sigma \{f_+(\sigma)\Omega_2(\sigma) + 2f_-(\sigma)\Omega_1(\sigma)\}, \quad (3.10)$$

where, $f_\pm(\sigma) = \frac{1}{2}(f(\sigma) \pm f(-\sigma))$ are by construction even and odd function and $f(\sigma)$ is an arbitrary differentiable function defined in the extended interval $[-\pi, \pi]$. The above expression can be simplified to:

$$\begin{aligned} L[f] &= \frac{1}{4} \int_{-\pi}^\pi d\sigma f(\sigma) [\Omega_2(\sigma) + 2\Omega_1(\sigma)] \\ &= \frac{1}{4} \int_{-\pi}^\pi d\sigma f(\sigma) [\Pi_\mu(\sigma) + X'_\mu(\sigma) - eB_{\mu\nu}X'^\nu(\sigma)]^2. \end{aligned} \quad (3.11)$$

It is now easy to verify (using (2.13)) that the above functional (3.11) generates the following Virasoro algebra:

$$\{L[f(\sigma)], L[g(\sigma)]\} = L[f(\sigma)g'(\sigma) - f'(\sigma)g(\sigma)]. \quad (3.12)$$

Defining

$$L_m = L[e^{-im\sigma}], \quad (3.13)$$

one can write down an equivalent form of the Virasoro algebra

$$\{L_m, L_n\} = i(m-n)L_{m+n}. \quad (3.14)$$

Note that we do not have a central extension here, as the analysis is entirely classical.

3.2 Interacting String in interpolating formalism

In the previous section we have reviewed the salient features of the interacting NG string. We now pass on to the construction of the interpolating action of the interacting string¹. To achieve this end, we write down the Lagrangian of the interacting NG action in the first-order form:

$$\mathcal{L}_I = \Pi_\mu \dot{X}^\mu - \mathcal{H}_T. \quad (3.15)$$

¹The construction of the interpolating action for the free string has been discussed in [10].

Substituting (3.7) in (3.15), \mathcal{L}_I becomes

$$\mathcal{L}_I = \Pi_\mu \dot{X}^\mu + \rho \Pi_\mu X'^\mu + \frac{\lambda}{2} \left[(\Pi^2 + X'^2) - 2eB_{\mu\nu} \Pi^\mu X'^\nu + e^2 B_{\mu\nu} B^\mu{}_\rho X'^\nu X'^\rho \right]. \quad (3.16)$$

The advantage of working with the interpolating action is that it naturally leads to either the NG or the Polyakov formulations of the string. In the Lagrangian (3.16), λ and ρ originally introduced as Lagrange multipliers, will be treated as independent fields, which behave as scalar and pseudo-scalar fields respectively in the extended world-sheet, as was discussed in the previous section. We will eliminate Π_μ from (3.16) as it is an auxiliary field. The Euler-Lagrange equation for Π_μ is:

$$\dot{X}^\mu + \rho X'^\mu + \lambda \Pi^\mu - e \lambda B^{\mu\nu} X'_\nu = 0. \quad (3.17)$$

Substituting Π_μ from (3.17) back in (3.16) yields:

$$\mathcal{L}_I = -\frac{1}{2\lambda} \left[\dot{X}^2 + 2\rho(\dot{X} \cdot X') + (\rho^2 - \lambda^2) X'^2 - 2\lambda e B_{\mu\nu} \dot{X}^\mu X'^\nu \right]. \quad (3.18)$$

This is the form of the interpolating Lagrangian of the interacting string.

The reproduction of the NG action (3.1) from the interpolating action of the interacting string is trivial and can be done by eliminating ρ and λ from their respective Euler-Lagrange equations of motion following from (3.18),

$$\begin{aligned} \rho &= -\frac{\dot{X} \cdot X'}{X'^2} \\ \lambda^2 &= \frac{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}{X'^2 X'^2}. \end{aligned} \quad (3.19)$$

From this equation λ is determined modulo a sign which can be fixed by demanding the consistency of Eq. (3.4) with Eq. (3.17). Accordingly,

$$\lambda = -\frac{\left[(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2 \right]^{\frac{1}{2}}}{X'^2 X'^2}. \quad (3.20)$$

If, on the other hand, we identify ρ and λ with the following contravariant components of the world-sheet metric,

$$g^{ab} = (-g)^{-\frac{1}{2}} \begin{pmatrix} \frac{1}{\lambda} & \frac{\rho}{\lambda} \\ \frac{\rho}{\lambda} & \frac{(\rho^2 - \lambda^2)}{\lambda} \end{pmatrix} \quad (3.21)$$

then the above Lagrangian (3.18) reduces to the Polyakov form,

$$\mathcal{L}_P = -\frac{1}{2} \left(\sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X_\mu - e \epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \right) \quad ; \quad (a, b = \tau, \sigma). \quad (3.22)$$

In this sense, therefore, the Lagrangian in (3.18) is referred to as an interpolating Lagrangian. It should be noted that the interpolating action has only two additional degrees of freedom, λ and ρ , which does not fully match the three degrees of freedom of the worldsheet metric of the Polyakov action. However, due to Weyl invariance of the Polyakov action, only two of the three different metric coefficients g_{ab} are really independent. This Weyl invariance is special to the Polyakov string, the higher branes do not share it.

We can now, likewise construct the interpolating BC from the interpolating Lagrangian (3.18),

$$K^\mu = \left[\frac{\partial \mathcal{L}_I}{\partial X'_\mu} \right]_{\sigma=0,\pi} = \left(\frac{\rho}{\lambda} \dot{X}^\mu + \frac{\rho^2 - \lambda^2}{\lambda} X'^\mu + e B^\mu{}_\nu \dot{X}^\nu \right)_{\sigma=0,\pi} = 0. \quad (3.23)$$

The fact that this can be easily interpreted as interpolating BC, can be easily seen by using the expressions (3.19) for ρ and λ in (3.23) to yield:

$$\left[\mathcal{L}_0^{-1} \left(-\dot{X}^2 X'^\mu + \left(\dot{X} X' \right) \dot{X}^\mu \right) - e B^{\mu\nu} \dot{X}_\nu \right]_{\sigma=0,\pi} = 0, \quad (3.24)$$

which is the BC of the interacting NG string (3.4).

On the other hand, we can identify ρ and λ with the metric components as in (3.21) to recast (3.23) as:

$$\left(g^{1a} \partial_a X^\mu(\sigma) + \frac{1}{\sqrt{-g}} e B^\mu{}_\nu \partial_0 X^\nu(\sigma) \right)_{\sigma=0,\pi} = 0. \quad (3.25)$$

which is easily identifiable with Polyakov form of BC [10] following from the action (3.22).

Using phase space variables X^μ and Π_μ , (3.23) can be rewritten as

$$K^\mu = \left[(\rho \Pi^\mu + \lambda X'^\mu) + e B^\mu{}_\nu \left(\Pi^\nu - e B^\nu{}_\rho X'^\rho \right) \right]_{\sigma=0,\pi} = 0. \quad (3.26)$$

Hence it is possible to interpret either of (3.23) or (3.26) as an interpolating BC.

Now we come to the discussion of the constraint structure of the interpolating interacting string. Note that the independent fields in (3.18) are X^μ , ρ and λ . The corresponding momenta denoted by Π_μ , π_ρ and π_λ , are given as:

$$\begin{aligned}\Pi_\mu &= -\frac{1}{\lambda} \left(\dot{X}_\mu + \rho X'_\mu \right) + e B_{\mu\nu} X'^\nu \\ \pi_\rho &= 0 \\ \pi_\lambda &= 0.\end{aligned}\tag{3.27}$$

In addition to the PB(s) similar to (2.12), we now have:

$$\begin{aligned}\{\rho(\tau, \sigma), \pi_\rho(\tau, \sigma')\} &= \delta(\sigma - \sigma') \\ \{\lambda(\tau, \sigma), \pi_\lambda(\tau, \sigma')\} &= \delta(\sigma - \sigma').\end{aligned}\tag{3.28}$$

The canonical Hamiltonian following from (3.18) reads:

$$\mathcal{H}_c = -\rho \Pi_\mu X'^\mu - \frac{\lambda}{2} \left\{ (\Pi_\mu - e B_{\mu\nu} X'^\nu)^2 + X'^2 \right\}\tag{3.29}$$

which reproduces the total Hamiltonian (3.7) of the NG action. From the definition of the canonical momenta we can easily identify the primary constraints:

$$\begin{aligned}\Omega_3 = \pi_\rho &= 0 \\ \Omega_4 = \pi_\lambda &= 0.\end{aligned}\tag{3.30}$$

The conservation of the above primary constraints leads to the secondary constraints Ω_1 and Ω_2 of (3.5). The primary constraints of the NG action appear as secondary constraints in this formalism. The system of constraints for the Interpolating Lagrangian thus comprises of the set (3.30) and (3.5). The PB(s) of the constraints of (3.30) vanish within themselves. Also the PB of these with (3.5) vanish.

3.3 Modified brackets for Interpolating String

3.3.1 Free Interpolating String:

Let us consider boundary condition for free interpolating string which can be obtained by setting $B_{\mu\nu} = 0$ in (3.26):

$$K^\mu = [(\rho\Pi^\mu + \lambda X'^\mu)]_{\sigma=0,\pi} = 0. \quad (3.31)$$

It is now easy to note that the above BC is not compatible with the basic PB (2.12). To incorporate this, an appropriate modification in the PB is in order. In the previous chapter, the equal time brackets were given in terms of certain combinations ($\Delta_+(\sigma, \sigma')$) of periodic delta function [10, 41, 50, 60],

$$\{X^\mu(\tau, \sigma), \Pi_\nu(\tau, \sigma')\} = \delta_\nu^\mu \Delta_+(\sigma, \sigma') \quad (3.32)$$

where,

$$\begin{aligned} \Delta_+(\sigma, \sigma') &= \delta_P(\sigma - \sigma') + \delta_P(\sigma + \sigma') = \frac{1}{\pi} + \frac{1}{\pi} \sum_{n \neq 0} \cos(n\sigma') \cos(n\sigma) \\ \Delta_-(\sigma, \sigma') &= \delta_P(\sigma - \sigma') - \delta_P(\sigma + \sigma') = \frac{1}{\pi} \sum_{n \neq 0} \sin(n\sigma') \sin(n\sigma) \end{aligned} \quad (3.33)$$

rather than an ordinary delta function to ensure compatibility with Neumann BC

$$\partial_\sigma X^\mu(\sigma)|_{\sigma=0,\pi} = 0, \quad (3.34)$$

in the bosonic sector. Remember that the other brackets

$$\{X^\mu(\sigma), X^\nu(\sigma')\} = 0 \quad (3.35)$$

$$\{\Pi^\mu(\sigma), \Pi^\nu(\sigma')\} = 0 \quad (3.36)$$

are consistent with the Neumann boundary condition (3.34).

Now a simple inspection shows that the BC (3.31) is also compatible with (3.32)² and (3.36), but not with (3.28) and (3.35). Hence the brackets (3.28) and (3.35) should be altered suitably.

²Note that there is no inconsistency in (3.34) as $\partial_\sigma \Delta_+(\sigma, \sigma')|_{\sigma=0,\pi} = 0$.

Now, since ρ and λ are odd and even functions of σ respectively, we propose:

$$\begin{aligned}\{\rho(\tau, \sigma), \pi_\rho(\tau, \sigma')\} &= \Delta_-(\sigma, \sigma') \\ \{\lambda(\tau, \sigma), \pi_\lambda(\tau, \sigma')\} &= \Delta_+(\sigma, \sigma').\end{aligned}\tag{3.37}$$

and also make the following ansatz for the bracket among the coordinates (3.35):

$$\{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} = C^{\mu\nu}(\sigma, \sigma') \quad ; \quad \text{where} \quad C^{\mu\nu}(\sigma, \sigma') = -C^{\nu\mu}(\sigma', \sigma).\tag{3.38}$$

One can easily check that the brackets (3.37) are indeed compatible with the BC (3.31). Now imposing the BC (3.31) on the above equation (3.38), we obtain the following condition:

$$\partial_\sigma C^{\mu\nu}(\sigma, \sigma')|_{\sigma=0, \pi} = \frac{\rho}{\lambda} \eta^{\mu\nu} \Delta_+(\sigma, \sigma')|_{\sigma=0, \pi}.\tag{3.39}$$

Now to find a solution for $C^{\mu\nu}(\sigma, \sigma')$, we choose³:

$$\partial_\sigma \left(\frac{\rho}{\lambda} \right) = 0\tag{3.40}$$

which gives a solution of $C^{\mu\nu}(\sigma, \sigma')$ as:

$$C^{\mu\nu}(\sigma, \sigma') = \eta^{\mu\nu} [\kappa(\sigma)\Theta(\sigma, \sigma') - \kappa(\sigma')\Theta(\sigma', \sigma)]\tag{3.41}$$

where the generalised step function $\Theta(\sigma, \sigma')$ satisfies,

$$\partial_\sigma \Theta(\sigma, \sigma') = \Delta_+(\sigma, \sigma').\tag{3.42}$$

Here, $\kappa(\sigma) = \frac{\rho}{\lambda}(\sigma)$ is a pseudo-scalar. The σ in the parenthesis has been included deliberately to remind the reader that it transforms as a pseudo-scalar under $\sigma \rightarrow -\sigma$ and should not be read as a functional dependence. The pseudo-scalar property of $\kappa(\sigma)$ is necessary for $C^{\mu\nu}(\sigma, \sigma')$ to be an even function of σ as $X(\sigma)$ is also an even function of σ in the extended interval $[-\pi, \pi]$ of the string (3.8). An explicit form of $\Theta(\sigma, \sigma')$ is given by [41]:

$$\Theta(\sigma, \sigma') = \frac{\sigma}{\pi} + \frac{1}{\pi} \sum_{n \neq 0} \frac{1}{n} \sin(n\sigma) \cos(n\sigma')\tag{3.43}$$

³The condition (3.40) reduces to a restricted class of metric for Polyakov formalism that satisfy $\partial_\sigma g_{01} = 0$. Such conditions also follow from a standard treatment of the light-cone gauge [53].

having the properties,

$$\begin{aligned}\Theta(\sigma, \sigma') &= 1 \quad \text{for } \sigma > \sigma' \\ \text{and } \Theta(\sigma, \sigma') &= 0 \quad \text{for } \sigma < \sigma'.\end{aligned}\tag{3.44}$$

Using the above relations, the simplified structure of (3.41) reads,

$$\begin{aligned}\{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} &= 0 \quad \text{for } \sigma = \sigma' \\ \{X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')\} &= \kappa(\sigma) \eta^{\mu\nu} \quad \text{for } \sigma > \sigma' \\ &= -\kappa(\sigma') \eta^{\mu\nu} \quad \text{for } \sigma < \sigma'.\end{aligned}\tag{3.45}$$

We therefore propose the brackets (3.32) and (3.45) as the basic PB(s) of the theory and using these one can easily obtain the following involutive algebra between the constraints:

$$\begin{aligned}\{\Omega_1(\sigma), \Omega_1(\sigma')\} &= \Omega_1(\sigma') \partial_\sigma \Delta_+(\sigma, \sigma') + \Omega_1(\sigma) \partial_\sigma \Delta_-(\sigma, \sigma') \\ \{\Omega_1(\sigma), \Omega_2(\sigma')\} &= (\Omega_2(\sigma) + \Omega_2(\sigma')) \partial_\sigma \Delta_+(\sigma, \sigma') \\ \{\Omega_2(\sigma), \Omega_2(\sigma')\} &= 4(\Omega_1(\sigma) \partial_\sigma \Delta_+(\sigma, \sigma') + \Omega_1(\sigma') \partial_\sigma \Delta_-(\sigma, \sigma')).\end{aligned}\tag{3.46}$$

The above algebra is exactly similar with the modified involutive algebra (2.35), between the constraints of the Polyakov theory.

We now compute the algebra between the Virasoro functionals using the modified constraint algebra (3.46),

$$\{L[f(\sigma)], L[g(\sigma)]\} = L[f(\sigma)g'(\sigma) - f'(\sigma)g(\sigma)].\tag{3.47}$$

Interestingly, the Virasoro algebra has the same form as that of (3.12) at the classical level. Consequently, the alternative forms of Virasoro algebra (3.14) is also reproduced here.

We shall study the consequences of the above algebra (3.46) in later section where we make an exhaustive analysis of gauge symmetry.

3.3.2 Interacting Interpolating String:

The Interpolating action for a bosonic string moving in the presence of a constant background Neveu-Schwarz two-form field $B_{\mu\nu}$ is given by,

$$S_I = \int d\tau d\sigma \left\{ -\frac{1}{2\lambda} \left[\dot{X}^2 + 2\rho(\dot{X} \cdot X') + (\rho^2 - \lambda^2) X'^2 - \lambda e \epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \right] \right\} \quad (3.48)$$

where $\epsilon^{01} = -\epsilon^{10} = +1$. The constraint structure has already been discussed in the section 3.

The BC (3.26) can be written in a completely covariant form as:

$$[M^\mu{}_\nu (\partial_\sigma X^\nu) + N^{\mu\nu} \Pi_\nu] |_{\sigma=0,\pi} = 0 \quad (3.49)$$

where,

$$\begin{aligned} M^\mu{}_\nu &= (\lambda \delta^\mu_\nu - e^2 B^{\mu\rho} B_{\rho\nu}) \\ N^{\mu\nu} &= (\rho \eta^{\mu\nu} + e B^{\mu\nu}). \end{aligned} \quad (3.50)$$

This nontrivial BC leads to a modification in the original (naive) PBs (2.12).

The BC (3.49) can be recast as:

$$(\partial_\sigma X^\mu + \Pi_\rho (NM^{-1})^{\rho\mu}) |_{\sigma=0,\pi} = 0. \quad (3.51)$$

The $\{X^\mu(\sigma), \Pi^\nu(\sigma')\}_{\text{PB}}$ is the same as that of the free string (3.32). We therefore make similar ansatz like (3.38) and using the BC (3.51), we get:

$$\partial_\sigma C_{\mu\nu}(\sigma, \sigma') |_{\sigma=0,\pi} = (NM^{-1})_{\nu\mu} \Delta_+(\sigma, \sigma') |_{\sigma=0,\pi}. \quad (3.52)$$

As in the free case, we restrict to the class defined by $\partial_\sigma (NM^{-1})_{\nu\mu} = 0$ which reduces to a restricted class of metric for Polyakov formalism. This reproduces the corresponding equation in interacting Polyakov string theory (see second chapter, in particular Eq 2.42). We therefore, obtain the following solution:

$$\begin{aligned} C_{\mu\nu}(\sigma, \sigma') &= \frac{1}{2} (NM^{-1})_{(\nu\mu)}(\sigma) \Theta(\sigma, \sigma') - \frac{1}{2} (NM^{-1})_{(\nu\mu)}(\sigma') \Theta(\sigma', \sigma) \\ &\quad + \frac{1}{2} (NM^{-1})_{[\nu\mu]}(\sigma) [\Theta(\sigma, \sigma') - 1] + \frac{1}{2} (NM^{-1})_{[\nu\mu]}(\sigma') \Theta(\sigma', \sigma) \end{aligned} \quad (3.53)$$

with $(NM^{-1})_{(\nu\mu)}$ the symmetric and $(NM^{-1})_{[\nu\mu]}$ the antisymmetric part of $(NM^{-1})_{\nu\mu}$.

3.4 Gauge symmetry

In this section we will discuss the gauge symmetries of the different actions and investigate their correspondence with the reparametrisation invariances. This has been done earlier for the free string case [66], however the canonical symplectic structure for the open string were not compatible with the general BC(s) of the theory. In the last two sections we have been discussing the BCs (3.23, 3.26) and have shown how the basic PB structures has to be modified suitably to be compatible with the BC(s). Now we shall investigate the gauge symmetry with the new modified PB structures (discussed in the earlier sections) which correctly takes into account the BC(s) of the theory. Importantly, the modified PB structure reveals a NC behavior among the string coordinates (3.41, 3.45). For simplicity the following analysis of the gauge symmetry is done for the case of the free strings.

All the constraints are first class and therefore generate gauge transformations on \mathcal{L}_I but the number of independent gauge parameters is equal to the number of independent primary first class constraints, i.e. two. In the following analysis we will apply a systematic procedure of abstracting the most general local symmetry transformations of the Lagrangian. A brief review of the procedure of [68, 69] will thus be appropriate.

Consider a theory with first class constraints only. The set of constraints Ω_a is assumed to be classified as

$$[\Omega_a] = [\Omega_{a_1}; \Omega_{a_2}] \quad (3.54)$$

where a_1 belong to the set of primary and a_2 to the set of secondary constraints. The total Hamiltonian is

$$H_T = H_c + \Sigma \lambda^{a_1} \Omega_{a_1} \quad (3.55)$$

where H_c is the canonical Hamiltonian and λ^{a_1} are Lagrange multipliers enforcing the primary constraints. The most general expression for the generator of gauge transformations is obtained according to the Dirac conjecture as

$$G = \Sigma \epsilon^a \Omega_a \quad (3.56)$$

where ϵ^a are the gauge parameters, only a_1 of which are independent. By demanding the

commutation of an arbitrary gauge variation with the total time derivative, (i.e. $\frac{d}{dt}(\delta q) = \delta\left(\frac{d}{dt}q\right)$) we arrive at the following equations [68, 72]

$$\delta\lambda^{a_1} = \frac{d\epsilon^{a_1}}{dt} - \epsilon^a \left(V_a^{a_1} + \lambda^{b_1} C_{b_1 a}^{a_1} \right) \quad (3.57)$$

$$0 = \frac{d\epsilon^{a_2}}{dt} - \epsilon^a \left(V_a^{a_2} + \lambda^{b_1} C_{b_1 a}^{a_2} \right) \quad (3.58)$$

Here the coefficients $V_a^{a_1}$ and $C_{b_1 a}^{a_1}$ are the structure functions of the involutive algebra, defined as

$$\begin{aligned} \{H_c, \Omega_a\} &= V_a^b \Omega_b \\ \{\Omega_a, \Omega_b\} &= C_{ab}^c \Omega_c. \end{aligned} \quad (3.59)$$

Solving (3.58) it is possible to choose a_1 independent gauge parameters from the set ϵ^a and express G of (3.56) entirely in terms of them. The other set (3.57) gives the gauge variations of the Lagrange multipliers.⁴

We begin the analysis with the interpolating Lagrangian (3.18). It contains additional fields ρ and λ . We shall calculate the gauge variation of these extra fields and explicitly show that they are connected to the reparametrization by a mapping between the gauge parameters and the diffeomorphism parameters. These maps will be obtained later in this section by demanding the consistency of the variations δX^μ due to gauge transformation and reparametrization

The full constraint structure of the theory comprises of the constraints (3.30) along with (3.5). We could proceed from these and construct the generator of gauge transformations. The generator of the gauge transformations of (3.18) is obtained by including the whole set of first class constraints Ω_i given by (3.30) and (3.5) as

$$G = \int d\sigma \alpha_i \Omega_i \quad (3.60)$$

where only two of the α_i 's are the independent gauge parameters. Using (3.58) the dependent gauge parameters could be eliminated. After finding the gauge generator in terms of the

⁴It can be shown that these equations are not independent conditions but appear as internal consistency conditions. In fact the conditions (3.57) follow from (3.58) [68, 69].

independent gauge parameters, the variations of the fields X^μ , ρ and λ can be worked out. But the number of independent gauge parameters are same in both NG (3.1) and interpolating (3.18) version. So the gauge generator⁵ is the same for both the cases, namely:

$$G = \int d\sigma (\alpha_1 \Omega_1 + \alpha_2 \Omega_2) \quad (3.61)$$

Also, looking at the intermediate first order form (3.16) it appears that the fields X^μ were already there in the NG action (3.1). The other two fields of the interpolating Lagrangian are ρ and λ which are nothing but the Lagrange multipliers enforcing the first class constraints (3.5) of the NG theory. Hence their gauge variation can be worked out from (3.57). We prefer to take this alternative route. For convenience we relabel ρ and λ by λ_1 and λ_2

$$\lambda_1 = \rho \quad \text{and} \quad \lambda_2 = \frac{\lambda}{2} \quad (3.62)$$

and their variations are obtained from (3.57)

$$\delta \lambda_i(\sigma) = -\dot{\alpha}_i - \int d\sigma' d\sigma'' C_{kj}{}^i(\sigma', \sigma'', \sigma) \lambda_k(\sigma') \alpha_j(\sigma'') \quad (3.63)$$

where $C_{kj}{}^i(\sigma', \sigma'', \sigma)$ are given by

$$\{\Omega_\alpha(\sigma), \Omega_\beta(\sigma')\} = \int d\sigma'' C_{\alpha\beta}{}^\gamma(\sigma, \sigma', \sigma'') \Omega_\gamma(\sigma'') \quad (3.64)$$

Observe that the structure function $V_a{}^b$ does not appear in (3.63) since $H_c = 0$ for the NG theory. The nontrivial structure functions $C_{\alpha\beta}{}^\gamma(\sigma, \sigma', \sigma'')$ are obtained from the constraint algebra (3.46) as:

$$\begin{aligned} C_{11}{}^1(\sigma, \sigma', \sigma'') &= (\partial_\sigma \Delta_+(\sigma, \sigma')) \Delta_-(\sigma', \sigma'') + (\partial_\sigma \Delta_-(\sigma, \sigma')) \Delta_-(\sigma, \sigma'') \\ C_{22}{}^1(\sigma, \sigma', \sigma'') &= 4(\partial_\sigma \Delta_+(\sigma, \sigma')) \Delta_-(\sigma, \sigma'') + 4(\partial_\sigma \Delta_-(\sigma, \sigma')) \Delta_-(\sigma', \sigma'') \\ C_{12}{}^2(\sigma, \sigma', \sigma'') &= \partial_\sigma \Delta_+(\sigma, \sigma') [\Delta_+(\sigma, \sigma'') + \Delta_+(\sigma', \sigma'')] \\ C_{21}{}^2(\sigma, \sigma', \sigma'') &= \partial_\sigma \Delta_-(\sigma, \sigma') [\Delta_+(\sigma, \sigma'') + \Delta_+(\sigma', \sigma'')] \end{aligned} \quad (3.65)$$

all other $C_{ab}{}^\gamma$'s are zero. Note that these structure functions are potentially different from those appearing in [66, 67] in the sense that here periodic delta functions are introduced to make

⁵Note that the gauge parameters α_1 and α_2 are odd and even respectively under $\sigma \rightarrow -\sigma$.

the basic brackets compatible with the nontrivial BC. Using the expressions of the structure functions (3.65) in equation (3.63) we can easily derive:

$$\begin{aligned}\delta\lambda_1 &= -\dot{\alpha}_1 + (\alpha_1\partial_1\lambda_1 - \lambda_1\partial_1\alpha_1) + 4(\alpha_2\partial_1\lambda_2 - \lambda_2\partial_1\alpha_2) \\ \delta\lambda_2 &= -\dot{\alpha}_2 + (\alpha_2\partial_1\lambda_1 - \lambda_1\partial_1\alpha_2) + (\alpha_1\partial_1\lambda_2 - \lambda_2\partial_1\alpha_1)\end{aligned}\quad (3.66)$$

From the correspondence (3.62), we get the variations of ρ and λ as:

$$\begin{aligned}\delta\rho &= -\dot{\alpha}_1 + (\alpha_1\partial_1\rho - \rho\partial_1\alpha_1) + 2(\alpha_2\partial_1\lambda - \lambda\partial_1\alpha_2) \\ \delta\lambda &= -2\dot{\alpha}_2 + 2(\alpha_2\partial_1\rho - \rho\partial_1\alpha_2) + (\alpha_1\partial_1\lambda - \lambda\partial_1\alpha_1)\end{aligned}\quad (3.67)$$

In the above we have found out the full set of symmetry transformations of the fields in the interpolating Lagrangian (3.18). These symmetry transformations (3.67) were earlier given in [70, 71] for the free string case. But the results were found there by inspection⁶. In our approach [66, 67] the appropriate transformations are obtained systematically by a general method applicable to a whole class of string actions.

We are still to investigate to what extent the exact correspondence between gauge symmetry and reparametrisation holds in our modified NC framework. This can be done very easily if we stick to the method discussed in [66, 67].

To work out the mapping between the gauge parameters and the diffeomorphism parameters we now take up the Polyakov action (3.22) (with $B = 0$). Here the only dynamic fields are X^μ . The transformations of X^μ under (3.61) can be worked out resulting in the following:

$$\delta X_\mu(\sigma) = \{X_\mu(\sigma), G\} = (\alpha_1 X'_\mu(\sigma) + 2\alpha_2 \Pi_\mu(\sigma)) \quad (3.68)$$

We can now substitute for Π_μ (obtained from (3.22)) to obtain:

$$\delta X_\mu = (\alpha_1 - 2\alpha_2 \sqrt{-g} g^{01}) X'_\mu - 2\sqrt{-g} g^{00} \alpha_2 \dot{X}_\mu \quad (3.69)$$

This is the gauge variation of X^μ in terms of X'_μ and \dot{X}_μ where the coefficients appear as arbitrary functions of σ and τ . So we can identify them with the arbitrary parameters Λ_1 and

⁶For easy comparison identify $\alpha_1 = \eta$ and $2\alpha_0 = \epsilon$

Λ_0 characterising the infinitesimal reparametrization [64]:

$$\begin{aligned}\tau' &= \tau - \Lambda_0 \\ \sigma' &= \sigma - \Lambda_1 \\ \delta X^\mu &= \Lambda^a \partial_a X^\mu = \Lambda_0 \dot{X}^\mu + \Lambda_1 X'^\mu\end{aligned}\tag{3.70}$$

and that of g_{ab} as:

$$\delta g_{ab} = D_a \Lambda_b + D_b \Lambda_a\tag{3.71}$$

where

$$D_a \Lambda_b = \partial_a \Lambda_b - \Gamma_{ab}^c \Lambda_c\tag{3.72}$$

Γ_{ab}^c being the usual Christoffel symbols [64]. The infinitesimal parameters Λ^a characterizes reparametrisation.

Comparing (3.69) and (3.70), we get the map connecting the gauge parameters with the diffeomorphism parameters:

$$\begin{aligned}\Lambda_0 &= -2\sqrt{-g} g^{00} \alpha_2 \\ \Lambda_1 &= \left(\alpha_1 - 2\alpha_2 \sqrt{-g} g^{01} \right)\end{aligned}\tag{3.73}$$

Using the definitions (3.21), this map can be cast in a better shape:

$$\begin{aligned}\Lambda_1 &= \left(\alpha_1 - 2\frac{\alpha_2 \rho}{\lambda} \right) \\ \Lambda_0 &= -\frac{2\alpha_2}{\lambda}\end{aligned}\tag{3.74}$$

All that remains now is to get the variation of ρ and λ induced by the reparametrisation (3.71). The identification (3.21) and (3.71) reproduces (3.67) as the variations of ρ and λ . This establishes complete equivalence of the gauge transformations with the diffeomorphisms of the string.

3.5 Summary

In this chapter, we have developed a new action formalism for interacting bosonic string and demonstrated that it interpolates between the NG and Polyakov form of interacting bosonic actions. This is similar to the interpolating action formalism for free string proposed in [10]. We have also modified the basic PBs in order to establish consistency of the BC with the basic PBs. We stress that contrary to standard approaches, BC(s) are not treated as primary constraints of the theory. Our approach is similar in spirit with the previous treatment of string theory [10, 41, 50, 60]. The NC structures derived in this chapter go over smoothly to the Polyakov version once suitable identifications are made. We then set out to study the status of gauge symmetries vis-à-vis reparametrisation in this NC set up and establish the connection between gauge symmetry and diffeomorphism transformations.

Chapter 4

Normal ordering and noncommutativity in open bosonic strings

So far we have discussed bosonic string theory classically now it would be quite interesting to study whether NC structure can be obtained from the conformal field theory techniques where the analysis is carried out in a quantum setting. Indeed, as has been stressed in [2] that it is very important to understand this noncommutativity from different perspectives. More importantly, it is necessary to check explicitly whether the central charge gets effected by the modified BCs in this case, as the central charge can be related to the Casimir energy arising from finite size of the string and is a purely quantum effect [53].

It may be recalled in this context that in quantum field theory, products of quantum fields at the same space-time points are in general singular objects. The same thing is true in string theory when one multiplies position operators, that can be taken as conformal fields on the world sheet. This situation is well known and one can remove the singular part of the operator products by defining normal ordered operators which have well behaved properties [53]. This is important, for example, when one builds up the generators of conformal transformations and investigates the realization of classical symmetries at quantum level.

Usually normal ordered products of operators are defined so as to satisfy the classical equations of motion at quantum level. However, in a recent paper [55], new normal ordered products have been defined for open string position operators that additionally satisfy the BCs. This way one obtains a normal ordering that is also valid at string end points. The mathematical problem posed by defining the normal ordering is related to that of calculating Green's functions [73, 74, 75, 76, 77, 78]. The normal ordered product is defined by subtracting out the corresponding Green's functions. So we can find normal ordered products satisfying open string boundary condition using the solutions to open string Green's functions.

In this chapter, we shall consider the problem of noncommutativity using this new normal ordering given in [55]. By using the contour argument and the new XX operator product expansion (OPE), we shall first find the commutator among the Fourier components and then the commutation relations among string's coordinates which reproduces the same noncommutative structure obtained in previous chapters and also in [10, 51], as mentioned earlier. We would also like to stress that the above commutator computed using the XX OPE satisfying the equation of motion only [53] (where the modifications due to BCs are not taken into account) also leads to the same NC structure.

4.1 New Normal ordered products

Let us first consider the free Polyakov string action,

$$S_P = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d\tau d\sigma \left(G^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \right) \quad (4.1)$$

where τ, σ are the usual world-sheet parameters, G_{ab} is the induced world-sheet metric with $G_{\tau\tau} = -1$, $G_{\sigma\sigma} = 1$ (upto a Weyl factor) in conformal gauge and the antisymmetric tensor is chosen by $\epsilon^{\tau\sigma} = 1$. $X^\mu(\tau, \sigma)$ are the string coordinates in the D dimensional Minkowskian target space with metric $\eta_{\mu\nu} = (-1, 1, \dots, 1)$.

We now make a Wick rotation by defining $\sigma^2 = i\tau$ and obtain the classical action for a

bosonic string taking a world-sheet with Euclidean signature:

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \left(g^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + i \varepsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \right) \quad (4.2)$$

where g^{ab} can now be taken to be proportional to the unit matrix and $\varepsilon^{12} = -\varepsilon^{21} = 1$. Note that the D dimensional target space-time still has the Lorentzian signature.

The variation of the action (4.2) gives the equation of motion,

$$(\partial_1^2 + \partial_2^2) X^\mu = 0 \quad (4.3)$$

and a boundary term that yields the following BCs:

$$\left(\partial_1 X^\mu(\sigma^1, \sigma^2) + i B_{\mu\nu} \partial_2 X^\nu(\sigma^1, \sigma^2) \right) |_{\sigma=0, \pi} = 0. \quad (4.4)$$

It is convenient, to introduce complex world sheet coordinates [53]: $z = \sigma^1 + i\sigma^2$; $\bar{z} = \sigma^1 - i\sigma^2$ and $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$.

In this notation the action is

$$S = \frac{1}{2\pi\alpha'} \int d^2z \left(\partial_z X^\mu \partial_{\bar{z}} X_\mu - B_{\mu\nu} \partial_z X^\mu \partial_{\bar{z}} X^\nu \right) \quad (4.5)$$

while the classical equations of motion and boundary conditions take the form

$$\partial_{\bar{z}} \partial_z X^\mu(z, \bar{z}) = 0 \quad (4.6)$$

$$\left[\eta^{\mu\nu} (\partial_z + \partial_{\bar{z}}) - B^{\mu\nu} (\partial_z - \partial_{\bar{z}}) \right] X_\nu |_{z=-\bar{z}, 2\pi-\bar{z}} = 0 \quad (4.7)$$

We now study the properties of quantum operators, corresponding to the classical variables, by considering the expectation values. Defining the expectation value of an operator \mathcal{F} as [53]:

$$\langle \mathcal{F}[X] \rangle = \int [dX] \exp(-S[X]) \mathcal{F}[X] \quad (4.8)$$

and using the fact that the path integral of a total derivative vanishes one finds:

$$\begin{aligned} 0 &= \int [dX] \frac{\delta}{\delta X^\nu(z', \bar{z}')} \exp(-S[X]) = \left\langle \frac{1}{\pi\alpha'} \partial_{\bar{z}'} \partial_{z'} X_\nu(z', \bar{z}') \right\rangle \\ &+ \frac{1}{2\pi\alpha'} \oint_{\partial\Sigma} \delta^2(z - z') \left\langle \left(\eta_{\nu\mu} (\partial_z + \partial_{\bar{z}}) + B_{\nu\mu} (\partial_z - \partial_{\bar{z}}) \right) X^\mu(z, \bar{z}) \right\rangle dz. \end{aligned} \quad (4.9)$$

The last (singular) term is integrated over the boundary, where $dz = -d\bar{z}$. We thus find that this equation implies that both string equations of motion and boundary condition hold as expectation values. So the corresponding quantum position operators \hat{X}^μ (in target space) satisfy (as long as they are not multiplied by other local operators coincident at the same world-sheet point) the following equations:

$$\partial_{\bar{z}}\partial_z\hat{X}^\nu(z, \bar{z}) = 0 \quad (4.10)$$

$$\left(\eta_{\nu\mu}(\partial_z + \partial_{\bar{z}}) - B_{\nu\mu}(\partial_z - \partial_{\bar{z}})\right)\hat{X}^\mu|_{\text{Bound.}} = 0 \quad (4.11)$$

which are nothing but the quantum version of (4.6, 4.7). Proceeding in the same way, we can consider a pair of local operators which may now be coincident to show that their products at the quantum level satisfy [55] :

$$\partial_{\bar{z}'}\partial_{z'}\hat{X}^\mu(z', \bar{z}')\hat{X}^\nu(z'', \bar{z}'') = -\pi\alpha'\eta^{\mu\nu}\delta^2(z' - z'', \bar{z}' - \bar{z}'') \quad (4.12)$$

$$\left(\eta_{\nu\mu}(\partial_{z'} + \partial_{\bar{z}'}) - B_{\nu\mu}(\partial_{z'} - \partial_{\bar{z}'})\right)\hat{X}^\mu(z', \bar{z}')\hat{X}^\rho(z'', \bar{z}'')|_{\text{Bound.}} = 0. \quad (4.13)$$

Now if we introduce the operation of normal ordering in the standard way [53],

$$\begin{aligned} : \hat{X}^\mu(z, \bar{z}) : &= \hat{X}^\mu(z, \bar{z}) \\ : \hat{X}^\mu(z, \bar{z}) \hat{X}^\nu(z', \bar{z}') : &= \hat{X}^\mu(z, \bar{z}) \hat{X}^\nu(z', \bar{z}') + \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z - z'|^2 \end{aligned} \quad (4.14)$$

it satisfies the equation of motion (4.6) at the quantum level, but fails to satisfy the boundary conditions (4.7). In [55], the authors have introduced a different kind of normal ordered product satisfying both equation of motion and boundary conditions.

At this point it is more convenient to choose world sheet coordinates, related to these z coordinates by conformal transformation, that simplify the representation of the boundary,

$$\omega = \exp(-iz) = e^{-i\sigma^1 + \sigma^2} ; \quad \bar{\omega} = e^{i\sigma^1 + \sigma^2}. \quad (4.15)$$

In this present coordinates the complete boundary corresponds just to the region $\omega = \bar{\omega}$. On the other hand, the action (4.5) along with equation of motion (4.10) in terms of $\omega, \bar{\omega}$ has still

the same form, while the form of BCs are slightly altered:

$$\partial_{\bar{\omega}} \partial_{\omega} \hat{X}^{\mu}(\omega, \bar{\omega}) = 0 \quad (4.16)$$

$$\left(\eta_{\mu\nu} (\partial_{\omega} - \partial_{\bar{\omega}}) - B_{\mu\nu} (\partial_{\omega} + \partial_{\bar{\omega}}) \right) \hat{X}^{\nu}|_{\omega=\bar{\omega}} = 0. \quad (4.17)$$

The corresponding new normal ordering is given by [55]:

$$\begin{aligned} : \hat{X}^{\mu}(\omega, \bar{\omega}) \hat{X}^{\nu}(\omega', \bar{\omega}') : &= \hat{X}^{\mu}(\omega, \bar{\omega}) \hat{X}^{\nu}(\omega', \bar{\omega}') + \frac{\alpha'}{2} \eta^{\mu\nu} \ln|\omega - \omega'|^2 + \frac{\alpha'}{2} \left([\eta - B]^{-1} [\eta + B] \right)^{\mu\nu} \\ &\quad \ln(\omega - \bar{\omega}') + \frac{\alpha'}{2} \left([\eta - B] [\eta + B]^{-1} \right)^{\mu\nu} \ln(\bar{\omega} - \omega') \end{aligned} \quad (4.18)$$

which satisfy both equation of motion and open string BCs (4.16, 4.17) at the quantum level.

These additional terms can be understood easily as ‘image’ contribution as in electrostatics.

Now for any arbitrary functional $\mathcal{F}[X]$, the new normal ordering (in absence of the B field) can be compactly written as:

$$:\mathcal{F}: = \exp \left(\frac{\alpha'}{4} \int d^2\omega_1 d^2\omega_2 \left[\ln|\omega_1 - \omega_2|^2 + \ln|\omega_1 - \bar{\omega}_2|^2 \right] \frac{\delta}{\delta X^{\mu}(\omega_1, \bar{\omega}_1)} \frac{\delta}{\delta X_{\mu}(\omega_2, \bar{\omega}_2)} \right) \mathcal{F}. \quad (4.19)$$

For example, this reproduces correctly the expression given in (4.18), as one can easily verify for $B = 0$.

The OPE for any pair of operators, satisfying the BCs, can be generated from

$$:\mathcal{F}::\mathcal{G}: = \exp \left(\frac{\alpha'}{4} \int d^2\omega_1 d^2\omega_2 \left[\ln|\omega_1 - \omega_2|^2 + \ln|\omega_1 - \bar{\omega}_2|^2 \right] \frac{\delta}{\delta X^{\mu}(\omega_1, \bar{\omega}_1)} \frac{\delta}{\delta X_{\mu}(\omega_2, \bar{\omega}_2)} \right) : \mathcal{F} \mathcal{G} : \quad (4.20)$$

It is now easy to verify that the TT OPE involving energy-momentum tensor

$$T(\omega) = -\frac{1}{\alpha'} : \partial X^{\mu}(\omega) \partial X_{\mu}(\omega) : \quad (4.21)$$

undergoes no modification. Indeed, using the above definition we obtain the following OPE:

$$\begin{aligned} : \partial X^{\mu}(\omega) \partial X_{\mu}(\omega) : : \partial' X^{\nu}(\omega') \partial' X_{\nu}(\omega') : &= : \partial X^{\mu}(\omega) \partial X_{\mu}(\omega) \partial' X^{\nu}(\omega') \partial' X_{\nu}(\omega') : \\ &\quad - 4 \cdot \frac{\alpha'}{2} \left(\partial \partial' \ln|\omega - \omega'|^2 \right) : \partial X^{\mu}(\omega) \partial' X_{\nu}(\omega') : \\ &\quad + 2 \cdot \delta^{\mu}_{\nu} \left(\frac{\alpha'}{2} \partial \partial' \ln|\omega - \omega'|^2 \right)^2 \\ &\sim \frac{D\alpha'^2}{2(\omega - \omega')^4} - \frac{2\alpha'}{(\omega - \omega')^2} : \partial' X^{\mu}(\omega') \partial' X_{\mu}(\omega') : \\ &\quad - \frac{2\alpha'}{(\omega - \omega')^2} \partial'^2 X^{\mu}(\omega') \partial' X_{\mu}(\omega') : \end{aligned} \quad (4.22)$$

where \sim mean ‘equal upto nonsingular terms’. The above result is same as that of [53] which is obtained by using the usual normal ordering satisfying the equation of motion only. This also implies that the Virasoro algebra remains the same as that of [53]. So the new normal ordering (4.18) (with $B = 0$) has no impact on the central charge.

We shall make use of the results discussed here in the next section where we study both free and interacting open bosonic strings.

4.2 Mode expansions and Non-Commutativity for bosonic strings

4.2.1 Free open strings

In this section, we consider the mode expansions of free ($B_{\mu\nu} = 0$) bosonic strings. We start with the closed string first. In the X^μ theory (4.5), ∂X^μ and $\bar{\partial} X^\mu$ are (anti)holomorphic and so have the following Laurent expansions,

$$\begin{aligned}\partial X^\mu(\omega) &= -i \left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_{m=-\infty}^{\infty} \frac{\alpha_m^\mu}{\omega^{m+1}} \\ \bar{\partial} X^\mu(\bar{\omega}) &= -i \left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_{m=-\infty}^{\infty} \frac{\tilde{\alpha}_m^\mu}{\bar{\omega}^{m+1}}.\end{aligned}\tag{4.23}$$

Now the BC(s) (4.17) in case of free open strings (i.e. $B_{\mu\nu} = 0$) requires $\alpha = \tilde{\alpha}$ in the expansions (4.23)¹. The expansion for X^μ is then:

$$X^\mu(\omega, \bar{\omega}) = x^\mu - i\alpha' p^\mu \ln|\omega|^2 + i \left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_{m \neq 0} \frac{\alpha_m^\mu}{m} (\omega^{-m} + \bar{\omega}^{-m})\tag{4.24}$$

where, x^μ and $p^\mu = \frac{1}{\sqrt{2\alpha'}} \alpha_0^\mu$ are the center of mass coordinate and momentum respectively.

Now the expressions (4.23) for open strings can be equivalently written as:

$$\alpha_m^\mu = \left(\frac{2}{\alpha'}\right)^{\frac{1}{2}} \oint \frac{d\omega}{2\pi} \omega^m \partial X^\mu(\omega) = \oint \frac{d\omega}{2\pi i} j_m^\mu(\omega)\tag{4.25}$$

¹Note that the BC(s) (4.17) in case of free open strings are the usual Neumann BC(s).

where, $j_m^\mu(\omega) = \sqrt{\frac{2}{\alpha'}} i \omega^m \partial X^\mu(\omega)$ is the corresponding holomorphic current. The commutation relation between α 's can be worked out from the contour argument and the $X X$ OPE² [53],

$$\begin{aligned} [\alpha_m^\mu, \alpha_n^\nu] &= \oint \frac{d\omega_2}{2\pi i} \text{Res}_{\omega_1 \rightarrow \omega_2} (j_m^\mu(\omega_1) j_n^\nu(\omega_2)) \\ &= m \delta_{m, -n} \eta^{\mu\nu} \end{aligned} \quad (4.26)$$

At this stage, it should be noted that the above approach does not give the algebra among the zero modes, i.e. the center of mass variables $[x^\mu, p^\nu]$ and $[x^\mu, x^\nu]$ of the open string. However, the results can be derived using standard techniques (as has been done in [51]) and read,

$$[x^\mu, p^\nu] = i \eta^{\mu\nu}. \quad (4.27)$$

The conjugate momenta $\Pi_\mu = \frac{1}{2\pi\alpha'} \dot{X}^\mu$ corresponding to X^μ can be calculated from the action (4.1) which has a Lorentzian signature for the world-sheet. In order to make a transition to Euclidean world-sheet, we make use of Wick rotation as before, by defining $\sigma^2 = i\tau$, so that $\dot{X}^\mu = i \frac{\partial X^\mu}{\partial \sigma^2}$. The $\Pi^\mu(\sigma^1, \sigma^2)$ can be recast as a function of ω and $\bar{\omega}$ using (4.15), so that its mode expansion becomes:

$$\Pi^\mu(\omega, \bar{\omega}) = \frac{1}{2\pi\alpha'} \left[2\alpha' p^\mu + \left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_{m \neq 0} \alpha_m^\mu (\omega^{-m} + \bar{\omega}^{-m}) \right] \quad (4.28)$$

where we have made use of the mode expansion of $X^\mu(\omega, \bar{\omega})$ (4.24). The commutation relations between $X^\mu(\omega, \bar{\omega})$ and $\Pi^\nu(\omega', \bar{\omega}')$ are then obtained by using (4.26, 4.27) as,

$$[X^\mu(\omega, \bar{\omega}), \Pi^\nu(\omega', \bar{\omega}')] = i \eta^{\mu\nu} \left(\frac{1}{\pi} + \frac{1}{4\pi} \sum_{m \neq 0} (\omega^{-m} + \bar{\omega}^{-m}) (\omega'^m + \bar{\omega}'^m) \right). \quad (4.29)$$

To obtain the usual equal time (i.e. $\tau = \tau'$) commutation relation we first rewrite (4.29) in “ z frame” using (4.15) and then in terms of σ^1, σ^2 to find,

$$[X^\mu(\sigma^1, \sigma^2), \Pi^\nu(\sigma^{1'}, \sigma^{2'})] = i \eta^{\mu\nu} \left[\frac{1}{\pi} + \frac{1}{\pi} \sum_{m \neq 0} \exp^{-m(\sigma^2 - \sigma^{2'})} \cos(m\sigma^1) \cos(m\sigma^{1'}) \right]. \quad (4.30)$$

²Note that here we have used the new normal ordering (4.18) for free open string, yet the commutation relations (4.26) remain same (see [53]).

Finally substituting $\tau = \tau'$ i.e. $\sigma^2 = \sigma'^2$ and $\sigma^1 = \sigma$ we get back the usual equal time commutation relation,

$$[X^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')] = i\eta^{\mu\nu} \Delta_+(\sigma, \sigma') \quad (4.31)$$

where,

$$\Delta_+(\sigma, \sigma') = \left[\frac{1}{\pi} + \frac{1}{\pi} \sum_{m \neq 0} \cos(m\sigma) \cos(m\sigma') \right]. \quad (4.32)$$

It is easy to see that (4.31) is compatible with Neumann BCs and reproduces the result in the previous chapters and [10, 41, 51].

4.2.2 Open string in the constant B-field background

We now analyse the open string moving in presence of a background Neveu-Schwarz two form field $B_{\mu\nu}$. To begin with, let us again consider the Laurent expansion of $\partial X^\mu(\omega)$ and $\bar{\partial} X^\mu(\bar{\omega})$ (4.23) for the case of closed string. But now the corresponding open string Laurent expansion is obtained by imposing the BCs, given in (4.17) with $B_{\mu\nu} \neq 0$ consequently, the modes α and $\tilde{\alpha}$ now satisfy:

$$\alpha_m^\mu - B^\mu_\nu \alpha_m^\nu = \tilde{\alpha}_m^\mu + B^\mu_\nu \tilde{\alpha}_m^\nu. \quad (4.33)$$

So there exists only one set of independent modes γ_m^μ , which can be thought of as the modes of free strings and is related to α_m^μ and $\tilde{\alpha}_m^\mu$ by:

$$\begin{aligned} \alpha_m^\mu &= (\delta^\mu_\nu + B^\mu_\nu) \gamma_m^\nu := [(\mathbb{1} + B)\gamma]_m^\mu \\ \tilde{\alpha}_m^\mu &= (\delta^\mu_\nu - B^\mu_\nu) \gamma_m^\nu := [(\mathbb{1} - B)\gamma]_m^\mu. \end{aligned} \quad (4.34)$$

Note that under world-sheet parity transformation $\alpha_m^\mu \leftrightarrow \tilde{\alpha}_m^\mu$, as $B_{\mu\nu}$ is a world-sheet pseudo-scalar. Substituting (4.34) in (4.23), we obtain the following Laurent expansions for ∂X^μ and $\bar{\partial} X^\mu$:

$$\begin{aligned} \partial X^\mu(\omega) &= -i \left(\frac{\alpha'}{2} \right)^{\frac{1}{2}} \sum_{m=-\infty}^{\infty} \frac{[(\mathbb{1} + B)\gamma]_m^\mu}{\omega^{m+1}} \\ \bar{\partial} X^\mu(\bar{\omega}) &= -i \left(\frac{\alpha'}{2} \right)^{\frac{1}{2}} \sum_{m=-\infty}^{\infty} \frac{[(\mathbb{1} - B)\gamma]_m^\mu}{\bar{\omega}^{m+1}}. \end{aligned} \quad (4.35)$$

Integrating the expansion (4.35) we obtain the mode expansion of $X^\mu(\omega, \bar{\omega})$ for the interacting string:

$$X^\mu(\omega, \bar{\omega}) = x^\mu - i\alpha' p^\mu \ln|\omega|^2 - i\alpha' B^\mu{}_\nu p^\nu (\ln\omega - \ln\bar{\omega}) + i\left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_{m \neq 0} \left[\frac{\gamma_m^\mu}{m} (\omega^{-m} + \bar{\omega}^{-m}) + \frac{1}{m} B^\mu{}_\nu \gamma_m^\nu (\omega^{-m} - \bar{\omega}^{-m}) \right]. \quad (4.36)$$

Now the expressions (4.35) for open interacting strings can also be equivalently written as:

$$\begin{aligned} [(\mathbb{1} + B)\gamma]_m^\mu &= \left(\frac{2}{\alpha'}\right)^{\frac{1}{2}} \oint \frac{d\omega}{2\pi} \omega^m \partial X^\mu(\omega) \\ [(\mathbb{1} - B)\gamma]_m^\mu &= -\left(\frac{2}{\alpha'}\right)^{\frac{1}{2}} \oint \frac{d\bar{\omega}}{2\pi} \bar{\omega}^m \partial X^\mu(\bar{\omega}). \end{aligned} \quad (4.37)$$

The commutation relation between γ 's can be obtained from the contour argument (using (4.37)) and the $X X$ OPE (4.18):

$$[\gamma_m^\mu, \gamma_n^\nu] = m \delta_{m, -n} \left[(\mathbb{1} - B^2)^{-1} \right]^{\mu\nu} = m \delta_{m, -n} (M^{-1})^{\mu\nu} \quad (4.38)$$

where, $M = (\mathbb{1} - B^2)$; $(B^2)^{\mu\nu} = B^\mu{}_\rho B^{\rho\nu}$ ³. Once again the algebra among the zero modes i.e. the center of mass variables $[x^\mu, p^\nu]$ and $[x^\mu, x^\nu]$ of the open string can not be obtained from the above contour arguments. The results can be derived using standard techniques (as discussed earlier) and read [51],

$$\begin{aligned} [x^\mu, p^\nu] &= i (M^{-1})^{\mu\nu} \\ [x^\mu, x^\nu] &= -2i \alpha' \pi (M^{-1} B)^{\mu\nu}. \end{aligned} \quad (4.39)$$

Now proceeding as in the free case, the conjugate momentum $\Pi^\mu(\omega, \bar{\omega})$ corresponding to (4.36) is:

$$\Pi^\mu(\omega, \bar{\omega}) = \frac{1}{2\pi\alpha'} \left[2\alpha' M^{\mu\rho} p_\rho + \left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_{m \neq 0} M^\mu{}_\rho \gamma_m^\rho (\omega^{-m} - \bar{\omega}^{-m}) \right]. \quad (4.40)$$

The commutators among the canonical variables $X^\mu(\omega, \bar{\omega}), \Pi^\nu(\omega, \bar{\omega})$ can be computed by using (4.38), (4.39),

$$[X^\mu(\omega, \bar{\omega}), \Pi^\nu(\omega', \bar{\omega}')] = i\eta^{\mu\nu} \left(\frac{1}{\pi} + \frac{1}{4\pi} \sum_{m \neq 0} (\omega^{-m} + \bar{\omega}^{-m}) (\omega'^m + \bar{\omega}'^m) \right)$$

³Here we should note that $(\mathbb{1})^{\mu\nu} = \eta^{\mu\nu}$.

$$\begin{aligned}
& + \frac{1}{4\pi} \sum_{m \neq 0} B^{\mu\nu} (\omega^{-m} - \bar{\omega}^{-m}) (\omega'^m + \bar{\omega}'^m) \Big) \quad (4.41) \\
[X^\mu(\omega, \bar{\omega}), X^\nu(\omega', \bar{\omega}')] &= \alpha' (M^{-1})^{\mu\nu} (\ln|\omega'|^2 - \ln|\omega|^2) - \alpha' (M^{-1}B)^{\mu\nu} \left(\ln \frac{\omega'}{\bar{\omega}'} + \ln \frac{\omega}{\bar{\omega}} + 2i\pi \right) \\
& + \frac{\alpha'}{2} \left(\sum_{m \neq 0} \frac{1}{m} [(M^{-1})^{\mu\nu} (\omega^{-m} + \bar{\omega}^{-m}) (\omega'^m + \bar{\omega}'^m) \right. \\
& + B^\mu{}_\rho B^\nu{}_\sigma (M^{-1})^{\mu\nu} (\omega^{-m} - \bar{\omega}^{-m}) (\omega'^m - \bar{\omega}'^m) \\
& - (M^{-1}B)^{\mu\nu} (\omega^{-m} + \bar{\omega}^{-m}) (\omega'^m - \bar{\omega}'^m) \\
& \left. + (M^{-1}B)^{\mu\nu} (\omega^{-m} + \bar{\omega}^{-m}) (\omega'^m + \bar{\omega}'^m) \right] \\
[\Pi^\mu(\omega, \bar{\omega}), \Pi^\nu(\omega', \bar{\omega}')] &= 0.
\end{aligned}$$

Now proceeding as before, we can rewrite the above commutation relation in σ^1, σ^2 coordinates to obtain the following,

$$\begin{aligned}
[X^\mu(\sigma^1, \sigma^2), \Pi^\nu(\sigma^{1'}, \sigma^{2'})] &= i\eta^{\mu\nu} \left(\frac{1}{\pi} + \frac{1}{\pi} \sum_{m \neq 0} \exp^{-m(\sigma^2 - \sigma^{2'})} [\cos(m\sigma^1) \cos(m\sigma^{1'}) \right. \\
& \left. + B^{\mu\nu} \sin(m\sigma^1) \cos(m\sigma^{1'}) \right] \Big) \quad (4.42) \\
[X^\mu(\sigma^1, \sigma^2), X^\nu(\sigma^{1'}, \sigma^{2'})] &= 2\alpha' (M^{-1})^{\mu\nu} (\sigma^{2'} - \sigma^2) + 2i\alpha' (M^{-1}B)^{\mu\nu} (\sigma^{1'} + \sigma^1 - \pi) \\
& + 2\alpha' \left(\sum_{m \neq 0} \frac{1}{m} e^{-m(\sigma^2 - \sigma^{2'})} [(M^{-1})^{\mu\nu} \cos(m\sigma^1) \cos(m\sigma^{1'}) \right. \\
& + B^\mu{}_\rho B^\nu{}_\sigma \sin(m\sigma^1) \sin(m\sigma^{1'}) \\
& \left. + i (M^{-1}B)^{\mu\nu} \sin(m(\sigma^1 + \sigma^{1'})) \right] \Big).
\end{aligned}$$

Finally we obtain the equal time commutation relations by identifying $\tau = \tau'$, i.e. $\sigma^2 = \sigma^{2'}$ and setting $\sigma^1 = \sigma$,

$$\begin{aligned}
[X^\mu(\tau, \sigma), \Pi^\nu(\tau, \sigma')] &= i\eta^{\mu\nu} \Delta_+(\sigma, \sigma') \\
[X^\mu(\tau, \sigma), X^\nu(\tau, \sigma')] &= 2i\alpha' (M^{-1}B)^{\mu\nu} \left[\sigma + \sigma' - \pi + \sum_{n \neq 0} \frac{1}{n} \sin(n(\sigma + \sigma')) \right]. \quad (4.43)
\end{aligned}$$

One can explicitly check that these commutators are compatible with BCs and also reproduces the result of 2nd chapter and ([7, 8, 10, 39, 51]).

4.3 Summary

In this chapter we have discussed an open bosonic string moving in the presence of a background Neveu-Schwarz two-form field $B_{\mu\nu}$ in a conformal field theory approach. We find the noncommutativity at the end point of the string. In contrast to several discussions, in which boundary conditions are taken as Dirac constraints, we have first obtained the mode algebra by using the contour argument and the newly proposed normal ordering, which satisfies both equations of motion and boundary conditions. Using these the commutator among the string coordinates is obtained. Interestingly, this new normal ordering yields the same algebra between the modes as the one satisfying only the equations of motion. In this approach, we find that noncommutativity originates more transparently and our results match with the noncommutative structure obtained in previous chapters and existing results in the literature.

Chapter 5

Superstring in a constant antisymmetric background field

We have already seen that open bosonic string, in presence of a back-ground Neveu-Schwarz two form field, leads to a noncommutative structure. In different approaches we have shown the same noncommutativity appears in the space-time coordinates of D -branes, where the end points of open string are attached.

The particular string theory described in this chapter is based on the introduction of a world-sheet supersymmetry that relates the space-time coordinates $X^\mu(\tau, \sigma)$, taken to be bosonic, to their fermionic counterparts $\psi^\mu(\tau, \sigma)$. The latter are two component world-sheet spinors. Then we study how non(anti) commutativity appears in the free super string and in a constant B-field background. This non(anti)commutativity is a direct consequence of the non trivial boundary conditions which, contrary to several approaches, are not treated as constraints. In this sense we have extended our methodology of bosonic theory to the superstring theory. We start with RNS superstring action in the conformal gauge. This also helps to fix the notations. Then we discuss the boundary conditions of the fermionic sector of the superstring and the non-anticommutativity of the theory.

5.1 Free superstring

Let us consider the action for the free superstring, in conformal gauge [79, 80],

$$S = \frac{i}{4} \int_{\Sigma} d^2\sigma d^2\theta (\overline{D}Y^\mu D Y_\mu), \quad (5.1)$$

where the superfield

$$Y^\mu(\sigma, \theta) = X^\mu(\sigma) + \bar{\theta}\psi^\mu(\sigma) + \frac{1}{2}\bar{\theta}\theta B^\mu(\sigma) \quad (5.2)$$

unites the bosonic ($X^\mu(\sigma)$) and fermionic ($\psi^\mu(\sigma)$) space-time string coordinates with a new auxiliary bosonic field $B^\mu(\sigma)$ whose utility may not be apparent at first. Note that both the bosonic (X^μ) and fermionic (ψ^μ) variables transform as vectors under target-space Lorentz transformations $SO(D-1, 1)$ and scalars under arbitrary world-sheet diffeomorphism, but as scalars and spinors, respectively, under world-sheet (local) Lorentz transformations $SO(1, 1)$ i.e. orthonormal transformation of tangent space at each point of the world-sheet.

Our signature of the induced world-sheet metric and target space-time metric are $\eta^{ab} = \{-, +\}$, $\eta^{\mu\nu} = \{-, +, +, \dots, +\}$ respectively and $\bar{\theta}$ is defined as $\bar{\theta} = \theta^T \rho^0$. The derivative

$$D_A = \frac{\partial}{\partial \theta^A} - i(\rho^a \theta)_A \partial_a$$

is known as the superspace covariant derivative. Here the symbol ρ^a represents two-dimensional Dirac matrices. A convenient basis is

$$\rho^0 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = i\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad (5.3)$$

they obey the Clifford algebra

$$\{\rho^a, \rho^b\} = -2\eta^{ab}. \quad (5.4)$$

The advantage of writing down the action in the superspace formalism, which includes the B^μ field, is that it is now manifestly invariant under infinitesimal supersymmetric transformations (with ϵ being infinitesimal Majorana spinor parameter)

$$\begin{aligned} \delta X^\mu &= \bar{\epsilon} \psi^\mu, \\ \delta \psi^\mu &= -i\rho^a \partial_a X^\mu \epsilon + B^\mu \epsilon, \\ \delta B^\mu &= -i\bar{\epsilon} \rho^a \partial_a \psi^\mu \end{aligned} \quad (5.5)$$

even without going on-shell. In absence of the B^μ field, however, one has to necessarily implement the on-shell condition.

In order to compute the θ integrals explicitly, we first note that

$$\begin{aligned} DY^\mu &= \psi^\mu + \theta B^\mu - i\rho^a \theta \partial_a X^\mu + \frac{i}{2} \bar{\theta} \theta \rho^a \partial_a \psi^\mu, \\ \bar{D}Y^\mu &= \bar{\psi}^\mu + B^\mu \bar{\theta} + i\partial_a X^\mu \bar{\theta} \rho^a - \frac{i}{2} \bar{\theta} \theta \partial_a \bar{\psi}^\mu \rho^a. \end{aligned} \quad (5.6)$$

It can easily be checked that both $D\Phi^\mu$ and $\bar{D}\Phi^\mu$ are invariant under SUSY transformations (5.5), so that SUSY invariance of the action (5.1) is manifest. In component form the action reads

$$\begin{aligned} S &= -\frac{1}{2} \int_\Sigma d^2\sigma \left(\eta_{\mu\nu} \partial_a X^\mu \partial^a X^\nu - i \bar{\psi}^\mu \rho^a \partial_a \psi_\mu \right) \\ &= S_B + S_F, \end{aligned} \quad (5.7)$$

where

$$\begin{aligned} S_B &= -\frac{1}{2} \int_\Sigma d^2\sigma \eta_{\mu\nu} \partial_a X^\mu \partial^a X^\nu, \\ S_F &= \frac{1}{2} \int_\Sigma d^2\sigma i \bar{\psi}^\mu \rho^a \partial_a \psi_\mu \end{aligned} \quad (5.8)$$

represent the decoupled bosonic and fermionic actions, respectively. The fermions are taken to be Majorana and we refer to the component of ψ in the basis (5.3) as ψ_\pm

$$\psi^\mu = \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix}, \quad (5.9)$$

which in the representation (5.3) are real (see appendix A).

The structure of the fermionic part S_F (5.8) of the action shows that the kinetic term is first order in the time derivative, consequently one can either employ the Dirac bracket formalism [36, 41] or the Faddeev-Jackiw method [52] to write down the following anti-bracket¹ :

$$\{\psi_A^\mu(\tau, \sigma), \psi_B^\nu(\tau, \sigma')\}_{D.B} = -i\eta^{\mu\nu} \delta_{AB} \delta(\sigma - \sigma'). \quad (5.10)$$

¹In this chapter, we have to clearly deal with a graded Lie-algebraic structure for the Poisson/Dirac brackets, but we shall make no distinction here on the notation.

The above antibrackets read, in terms of the components of ψ ,

$$\begin{aligned}\{\psi_+^\mu(\sigma), \psi_+^\nu(\sigma')\}_{D.B} &= \{\psi_-^\mu(\sigma), \psi_-^\nu(\sigma')\}_{D.B} = -i\eta^{\mu\nu}\delta(\sigma - \sigma'), \\ \{\psi_+^\mu(\sigma), \psi_-^\nu(\sigma')\}_{D.B} &= 0.\end{aligned}\tag{5.11}$$

This, along with the brackets

$$\{X^\mu(\sigma), \Pi^\nu(\sigma')\} = \eta^{\mu\nu}\delta(\sigma - \sigma')\tag{5.12}$$

from the bosonic sector, defines the preliminary symplectic structure of the theory (Π^μ is the canonically conjugate momentum to X^μ , defined in the usual way).

Confining our attention to S_F , we vary the action (5.8)

$$\delta S_F = i \int_{\Sigma} d^2\sigma \left[\rho^a \partial_a \psi^\mu \delta \bar{\psi}_\mu - \partial_\sigma (\psi_-^\mu \delta \psi_{\mu-} - \psi_+^\mu \delta \psi_{\mu+}) \right]\tag{5.13}$$

to obtain the Euler-Lagrange equation for fermionic field

$$i\rho^a \partial_a \psi^\mu = 0,\tag{5.14}$$

which further reduces to

$$\left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \sigma} \right) \psi_-^\mu = 0 \quad ; \quad \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \sigma} \right) \psi_+^\mu = 0.\tag{5.15}$$

This indicates that the functional dependence of fermi-fields are given by $\psi_{\mp}(\tau \mp \sigma)$. The total divergence term yields the necessary BC. We shall consider its consequences in the following sections where the preliminary (anti) brackets will be modified. The action S of (5.7) is invariant (using equation of motion) under the infinitesimal transformations

$$\begin{aligned}\delta X^\mu &= \bar{\epsilon} \psi^\mu \\ \delta \psi^\mu &= -i\rho^a \partial_a X^\mu \epsilon,\end{aligned}\tag{5.16}$$

with ϵ a constant anticommuting spinor. One can generate these transformations through the generator

$$Q_A = \frac{\partial}{\partial \theta^A} + i(\rho^a \theta)_A \partial_a.\tag{5.17}$$

The super charge Q is used to define transformations of the coordinates

$$\delta Y^\mu = \bar{\epsilon} Q Y^\mu \quad (5.18)$$

Since $\{D_A, Q_B\} = 0$ i.e. derivative operator is invariant under supersymmetry, the action (5.7) is also invariant under supersymmetry. Using the standard Noether procedure², the forms of the supercurrent and the energy-momentum tensor (which are constraints themselves [79]) can be derived. The expressions are:

$$J_a = -\frac{1}{2}\rho^b \rho_a \psi^\mu \partial_b X_\mu = 0, \quad (5.19)$$

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{i}{4}\bar{\psi}^\mu \rho_a \partial_b \psi_\mu - \frac{i}{4}\bar{\psi}^\mu \rho_b \partial_a \psi_\mu - \frac{1}{2}\eta_{ab}(\partial^c X^\mu \partial_c X_\mu + \frac{i}{2}\bar{\psi}^\mu \rho^a \partial_a \psi_\mu) = 0. \quad (5.20)$$

All the components of T_{ab} are, however, not independent as the energy-momentum tensor is traceless

$$T^a_a = \eta^{ab} T_{ab} = 0, \quad (5.21)$$

leaving us with only two independent components of T_{ab} . These components, which are the constraints of the theory, are given by

$$\begin{aligned} \chi_1(\sigma) = 2T_{00} = 2T_{11} &= \Omega_1(\sigma) + \lambda_1(\sigma) = 0 \\ \chi_2(\sigma) = T_{01} &= \Omega_2(\sigma) + \lambda_2(\sigma) = 0. \end{aligned} \quad (5.22)$$

where,

$$\begin{aligned} \Omega_1(\sigma) &= (\Pi^2(\sigma) + (\partial_\sigma X(\sigma))^2) \\ \Omega_2(\sigma) &= (\Pi(\sigma) \partial_\sigma X(\sigma)) \\ \lambda_1(\sigma) &= -i\bar{\psi}^\mu(\sigma) \rho_1 \partial_\sigma \psi_\mu(\sigma) = -i(\psi_-^\mu(\sigma) \partial_\sigma \psi_{\mu-}(\sigma) - \psi_+^\mu(\sigma) \partial_\sigma \psi_{\mu+}(\sigma)) \\ \lambda_2(\sigma) &= -\frac{i}{2}\bar{\psi}^\mu(\sigma) \rho_0 \partial_\sigma \psi_\mu(\sigma) = \frac{i}{2}(\psi_-^\mu(\sigma) \partial_\sigma \psi_{\mu-}(\sigma) + \psi_+^\mu(\sigma) \partial_\sigma \psi_{\mu+}(\sigma)). \end{aligned} \quad (5.23)$$

²We now use the supersymmetry transformations on-shell and hence we drop the auxiliary field B^μ henceforth.

The role of these constraints in generating infinitesimal diffeomorphisms is well known [79, 80] and we are not going to elaborate on this. Note that the constraints that we obtain in this chapter are on-shell, i.e. we have used the equation of motion (5.14) for the fermionic field ψ . This allows us to write them down in terms of the phase-space variables³ and hence they look quite different from the standard results found in the literature [79, 80] where they are written down in the light-cone coordinates which involves time derivatives.

From the basic brackets (5.11), it is easy to generate a first class (involutive) algebra,

$$\begin{aligned}\{\chi_1(\sigma), \chi_1(\sigma')\} &= 4(\chi_2(\sigma) + \chi_2(\sigma')) \partial_\sigma \delta(\sigma - \sigma') \\ \{\chi_2(\sigma), \chi_2(\sigma')\} &= (\chi_2(\sigma) + \chi_2(\sigma')) \partial_\sigma \delta(\sigma - \sigma') \\ \{\chi_2(\sigma), \chi_1(\sigma')\} &= (\chi_1(\sigma) + \chi_1(\sigma')) \partial_\sigma \delta(\sigma - \sigma').\end{aligned}\tag{5.24}$$

It is interesting to observe that the structure of the super first class constraint algebra is exactly similar to that of the constraint algebra (2.13) of Bosonic theory.

Coming to the super current J_{aA} ⁴, note that it is a two component spinor. Further, since J_a obeys the relation $\rho^a J_a = 0$, the components of J_{0A} and J_{1A} are related to each other. Hence we only deal with the components of J_{0A} or simply J_1 and J_2 . These J_1, J_2 along with $\chi_1(\sigma)$ and $\chi_2(\sigma)$ constitutes the full set of super-Virasoro constraints. For convenience we write:

$$\begin{aligned}\tilde{J}_1(\sigma) &= 2J_1(\sigma) = (\psi_-^\mu(\sigma) \Pi_\mu(\sigma) - \psi_-^\mu(\sigma) \partial_\sigma X_\mu) = 0, \\ \tilde{J}_2(\sigma) &= 2J_2(\sigma) = (\psi_+^\mu(\sigma) \Pi_\mu(\sigma) + \psi_+^\mu(\sigma) \partial_\sigma X_\mu) = 0.\end{aligned}\tag{5.25}$$

The algebra between the above constraints read:

$$\begin{aligned}\{\tilde{J}_1(\sigma), \tilde{J}_1(\sigma')\} &= -i(\chi_1(\sigma) - 2\chi_2(\sigma)) \delta(\sigma - \sigma'), \\ \{\tilde{J}_2(\sigma), \tilde{J}_2(\sigma')\} &= -i(\chi_1(\sigma) + 2\chi_2(\sigma)) \delta(\sigma - \sigma'), \\ \{\tilde{J}_1(\sigma), \tilde{J}_2(\sigma')\} &= 0.\end{aligned}\tag{5.26}$$

The algebra between $\tilde{J}(\sigma)$ and $\chi(\sigma)$ is also given by

$$\{\chi_1(\sigma), \tilde{J}_1(\sigma')\} = -(2\tilde{J}_1(\sigma) + \tilde{J}_1(\sigma')) \partial_\sigma \delta(\sigma - \sigma'),$$

³This is in the true spirit of Dirac's classic analysis of constrained hamiltonian dynamics [36].

⁴ $A = 1, 2$ being the spinor index

$$\begin{aligned}
\{\chi_1(\sigma), \tilde{J}_2(\sigma')\} &= \left(2\tilde{J}_2(\sigma) + \tilde{J}_2(\sigma')\right) \partial_\sigma \delta(\sigma - \sigma'), \\
\{\chi_2(\sigma), \tilde{J}_1(\sigma')\} &= \left(\tilde{J}_1(\sigma) + \frac{1}{2}\tilde{J}_1(\sigma')\right) \partial_\sigma \delta(\sigma - \sigma'), \\
\{\chi_2(\sigma), \tilde{J}_2(\sigma')\} &= \left(\tilde{J}_2(\sigma) + \frac{1}{2}\tilde{J}_2(\sigma')\right) \partial_\sigma \delta(\sigma - \sigma').
\end{aligned} \tag{5.27}$$

5.2 Boundary conditions and super-Virasoro algebra for superstring

As in the case of bosonic variables, Fermionic coordinates also require careful consideration of the surface terms arising in the variation of the action (5.13). Vanishing of these surface terms requires that $(\psi_+ \delta\psi_+ - \psi_- \delta\psi_-)$ should vanish at each end point of the open string. This is satisfied by making $\psi_+ = \pm\psi_-$ at each end. Without loss of generality we set

$$\psi_+^\mu(0, \tau) = \psi_-^\mu(0, \tau). \tag{5.28}$$

The relative sign at the other end now becomes meaningful and there are two cases to be considered. In the first case (Ramond(R) boundary conditions)

$$\psi_+^\mu(\pi, \tau) = \psi_-^\mu(\pi, \tau) \tag{5.29}$$

and in the second case (Neveu-Schwarz (NS) boundary conditions)

$$\psi_+^\mu(\pi, \tau) = -\psi_-^\mu(\pi, \tau). \tag{5.30}$$

Here we will work with Ramond boundary conditions and in the last chapter we shall discuss the Neveu-Schwarz boundary conditions in detail. Combining (5.28) and (5.29) we can write

$$\left(\psi_+^\mu(\tau, \sigma) - \psi_-^\mu(\tau, \sigma)\right) |_{\sigma=0, \pi} = 0. \tag{5.31}$$

As discussed in the appendix, we have $\psi_L(\tau - \sigma) = -i\psi_R(\tau + \sigma)$ in the chiral representation (see (A.15) in appendix), which translates into

$$\psi_-^\mu(-\sigma, \tau) = \psi_+^\mu(\sigma, \tau) \tag{5.32}$$

in the representation (5.3). Using the BC (5.29), we can therefore write

$$\psi_{\pm}^{\mu}(\sigma = \pi, \tau) = \psi_{\pm}^{\mu}(\sigma = -\pi, \tau) \quad (5.33)$$

in R-sector. We can then extend the range of definition of ψ_{\pm}^{μ} from $[0, \pi]$ to $[-\pi, \pi]$ with periodic BC imposed on ψ_{\pm}^{μ} of period 2π . Consequently, the mode expansion of the components of Majorana fermion takes the form

$$\begin{aligned} \psi_{-}^{\mu}(\sigma, \tau) &= \frac{1}{\sqrt{2}} \sum d_n^{\mu} e^{-in(\tau-\sigma)}, \\ \psi_{+}^{\mu}(\sigma, \tau) &= \frac{1}{\sqrt{2}} \sum d_n^{\mu} e^{-in(\tau+\sigma)}. \end{aligned} \quad (5.34)$$

On the other hand, in chapter 2 we have already enlarged the domain of definition of the bosonic field X^{μ} as

$$X^{\mu}(\tau, -\sigma) = X^{\mu}(\tau, \sigma) \quad (5.35)$$

so that it is an even function and satisfies Neumann boundary condition. This is in contrast to (5.32). Consistent with this, we also have

$$\begin{aligned} \Pi^{\mu}(\tau, -\sigma) &= \Pi^{\mu}(\tau, \sigma) \\ X^{\mu'}(\tau, -\sigma) &= -X^{\mu'}(\tau, \sigma). \end{aligned} \quad (5.36)$$

Now from (5.32), we note that the constraints $\chi_1(\sigma) = 0$ and $\chi_2(\sigma) = 0$ are even and odd respectively under $\sigma \rightarrow -\sigma$. This also enables us to increase the domain of definition of the length of the string from $(0 \leq \sigma \leq \pi)$ to $(-\pi \leq \sigma \leq \pi)$. We may then write the generator of all τ and σ reparametrization as the functional [41]

$$L[f] = \frac{1}{2} \int_0^{\pi} d\sigma \{f_{+}(\sigma)\chi_1(\sigma) + 2f_{-}(\sigma)\chi_2(\sigma)\}, \quad (5.37)$$

where $f_{\pm}(\sigma) = \frac{1}{2}(f(\sigma) \pm f(-\sigma))$ are by construction even/odd function and $f(\sigma)$ is an arbitrary differentiable function defined in the extended interval $[-\pi, \pi]$. The above expression can be simplified to

$$L[f] = \frac{1}{4} \int_{-\pi}^{\pi} d\sigma f(\sigma) [\{\Pi(\sigma) + \partial_{\sigma} X(\sigma)\}^2 + 2i\psi_{+}^{\mu} \partial_{\sigma} \psi_{\mu+}]. \quad (5.38)$$

Coming to the generators J_1 and J_2 , note that $J_1(-\sigma) = J_2(\sigma)$ (5.25). This enables us to write down the functional $G[g]$

$$\begin{aligned} G[g] &= \int_0^\pi d\sigma (g(\sigma)J_1(\sigma) + g(-\sigma)J_2(\sigma)) \\ &= \int_{-\pi}^\pi d\sigma g(\sigma)J_1(\sigma) = \int_{-\pi}^\pi d\sigma g(-\sigma)J_2(\sigma) \end{aligned} \quad (5.39)$$

for any differentiable function $g(\sigma)$, defined again in the extended interval $[-\pi, \pi]$. These functionals (5.38), (5.39) generate the following super Virasoro algebra

$$\begin{aligned} \{L[f(\sigma)], L[g(\sigma)]\} &= L[f(\sigma)g'(\sigma) - f'(\sigma)g(\sigma)], \\ \{G[g(\sigma)], G[h(\sigma)]\} &= -iL[g(-\sigma)h(-\sigma)], \\ \{L[f(\sigma)], G[g(\sigma)]\} &= G[f(\sigma)g'(-\sigma) - \frac{1}{2}f'(\sigma)g(-\sigma)]. \end{aligned} \quad (5.40)$$

Defining

$$L_m = L[e^{-im\sigma}] \quad \text{and} \quad G_n = G[e^{in\sigma}], \quad (5.41)$$

one can write down an equivalent form of the super-Virasoro algebra

$$\begin{aligned} \{L_m, L_n\} &= i(m-n)L_{m+n}, \\ \{G_m, G_n\} &= -iL_{m+n}, \\ \{L_m, G_n\} &= i\left(\frac{m}{2} - n\right)G_{m+n}. \end{aligned} \quad (5.42)$$

Note that we do not have a central extension here, as the analysis is entirely classical.

5.3 Non(anti)commutativity for open superstrings

Coming back to the preliminary symplectic structure, given in (5.11), we note that the boundary conditions (5.31) are not compatible with the brackets, although one could get the super-Virasoro algebra (5.40) or (5.42) just by using (5.11) and (5.12). Hence the last of the brackets in (5.11) should be altered suitably. A simple inspection suggests that

$$\{\psi_+^\mu(\sigma), \psi_-^\nu(\sigma')\} = -i\eta^{\mu\nu}\delta(\sigma - \sigma'). \quad (5.43)$$

Although the bracket structures (5.11) and (5.43) agree with [58] (in the free case), they can, however, not be regarded as the final ones. This is because the presence of the usual Dirac delta function $\delta(\sigma - \sigma')$ implicitly implies that the finite physical range of $\sigma \in [0, \pi]$ for the string has not been taken into account. Besides, it is also not compatible with (5.32). Further, the anti-brackets (5.43) are a bit naive simply because the support of σ in the above (usual) delta function, or more precisely a distribution, is $[-\infty, +\infty]$ and is not compatible with the compact support of σ in the physical range of the string which is $[0, \pi]$. This motivates us to modify the above anti-brackets suitably. However, in order to combine the fermionic and the bosonic sectors, one need to modify the above antibrackets. In the previous chapters, the equal time commutators were given in terms of $\Delta_+(\sigma, \sigma')$, a certain combinations of periodic delta function

$$\{X^\mu(\tau, \sigma), \Pi_\nu(\tau, \sigma')\} = \delta_\nu^\mu \Delta_+(\sigma, \sigma'), \quad (5.44)$$

where

$$\Delta_\pm(\sigma, \sigma') = \delta_P(\sigma - \sigma') \pm \delta_P(\sigma + \sigma'), \quad (5.45)$$

rather than an ordinary delta function to ensure compatibility with Neumann BC in the bosonic sector. Basically, there one has to identify the appropriate “delta function” for the physical range $[0, \pi]$ of σ starting from the periodic delta function $\delta_P(\sigma - \sigma')$ for the extended (but finite) range $[-\pi, \pi]$ and make use of the even nature of the bosonic variables X^μ in the extended interval.

We can essentially follow the same methodology here in the fermionic sector as $\psi_\pm^\mu(\tau, \sigma)$ also satisfy periodic BC of period 2π (5.33). The only difference with the bosonic case, apart from the Grassmanian nature of the latter, is that, instead of their even property (5.35), the components of Majorana fermions satisfy (5.32). As we shall show now this condition is quite adequate to identify the appropriate delta-functions for the “physical interval” $[0, \pi]$.

We start by noting that the usual properties of a delta function is also satisfied by $\delta_P(x)$

$$\int_{-\pi}^{\pi} dx' \delta_P(x' - x) f(x') = f(x) \quad (5.46)$$

for any periodic function $f(x) = f(x + 2\pi)$ defined in the interval $[-\pi, \pi]$. Hence one can immediately write down the following expressions for ψ_-^μ and ψ_+^μ :

$$\int_0^\pi d\sigma' [\delta_P(\sigma' + \sigma)\psi_+^\mu(\sigma') + \delta_P(\sigma' - \sigma)\psi_-^\mu(\sigma')] = \psi_-^\mu(\sigma) \quad (5.47)$$

$$\int_0^\pi d\sigma' [\delta_P(\sigma' + \sigma)\psi_-^\mu(\sigma') + \delta_P(\sigma' - \sigma)\psi_+^\mu(\sigma')] = \psi_+^\mu(\sigma). \quad (5.48)$$

Combining the above equations and writing them in a matrix form, we get,

$$\int_0^\pi d\sigma' \Lambda_{AB}(\sigma, \sigma') \psi_B^\mu(\sigma') = \psi_A^\mu(\sigma) \quad ; \quad (A = -, +), \quad (5.49)$$

where $\Lambda_{AB}(\sigma, \sigma')$, defined by

$$\Lambda_{AB}(\sigma, \sigma') = \begin{pmatrix} \delta_P(\sigma' - \sigma) & \delta_P(\sigma' + \sigma) \\ \delta_P(\sigma' + \sigma) & \delta_P(\sigma' - \sigma) \end{pmatrix}, \quad (5.50)$$

acts like a matrix valued “delta function” for the two component Majorana spinor in the reduced physical interval $[0, \pi]$ of the string. We therefore propose the following anti-brackets in the fermionic sector:

$$\{\psi_A^\mu(\sigma), \psi_B^\nu(\sigma')\} = -i\eta^{\mu\nu} \Lambda_{AB}(\sigma, \sigma'), \quad (5.51)$$

instead of (5.10) which, when written down explicitly in terms of components, reads

$$\begin{aligned} \{\psi_+^\mu(\sigma), \psi_+^\nu(\sigma')\} &= \{\psi_-^\mu(\sigma), \psi_-^\nu(\sigma')\} = -i\eta^{\mu\nu} \delta_P(\sigma - \sigma'), \\ \{\psi_-^\mu(\sigma), \psi_+^\nu(\sigma')\} &= -i\eta^{\mu\nu} \delta_P(\sigma + \sigma'). \end{aligned} \quad (5.52)$$

We shall now investigate the consistency of this structure. Firstly, this structure of the anti-bracket relations is completely consistent with the boundary condition (5.31). To see this explicitly, we compute the anticommutator of $\psi_+(\sigma')$ with (5.31), the left hand side of which gives

$$\begin{aligned} -i(\delta_P(\sigma - \sigma') - \delta_P(\sigma + \sigma'))|_{\sigma=0, \pi} &= -i\Delta_-(\sigma, \sigma')|_{\sigma=0, \pi} \\ &= \frac{1}{\pi} \sum_{n \neq 0} \sin(n\sigma') \sin(n\sigma)|_{\sigma=0, \pi} = 0, \end{aligned} \quad (5.53)$$

where the form of the periodic delta function has been used. Not only that, as a bonus, we reproduce the modified form of (5.43). Observe the occurrence of $\delta_P(\sigma + \sigma')$ rather than

$\delta_P(\sigma - \sigma')$ in the mixed bracket $\{\psi_+, \psi_-\}$, which plays a crucial role in obtaining the following involutive algebra in the fermionic sector⁵. Indeed, using (5.51), one can show that

$$\begin{aligned}\{\lambda_1(\sigma), \lambda_1(\sigma')\} &= 4(\lambda_2(\sigma)\partial_\sigma\Delta_+(\sigma, \sigma') + \lambda_2(\sigma')\partial_\sigma\Delta_-(\sigma, \sigma')) , \\ \{\lambda_2(\sigma), \lambda_2(\sigma')\} &= (\lambda_2(\sigma')\partial_\sigma\Delta_+(\sigma, \sigma') + \lambda_2(\sigma)\partial_\sigma\Delta_-(\sigma, \sigma')) , \\ \{\lambda_2(\sigma), \lambda_1(\sigma')\} &= (\lambda_1(\sigma) + \lambda_1(\sigma'))\partial_\sigma\Delta_+(\sigma, \sigma')\end{aligned}\tag{5.54}$$

hold for the fermionic sector.

Remarkably the above constraint algebra is exactly similar to the constraint algebra of the bosonic sector (2.35). This helps us to write down the complete algebra of the super Virasoro constraints $\chi_1(\sigma)$ and $\chi_2(\sigma)$:

$$\begin{aligned}\{\chi_1(\sigma), \chi_1(\sigma')\} &= 4(\chi_2(\sigma)\partial_\sigma\Delta_+(\sigma, \sigma') + \chi_2(\sigma')\partial_\sigma\Delta_-(\sigma, \sigma')) , \\ \{\chi_2(\sigma), \chi_2(\sigma')\} &= (\chi_2(\sigma')\partial_\sigma\Delta_+(\sigma, \sigma') + \chi_2(\sigma)\partial_\sigma\Delta_-(\sigma, \sigma')) , \\ \{\chi_2(\sigma), \chi_1(\sigma')\} &= (\chi_1(\sigma) + \chi_1(\sigma'))\partial_\sigma\Delta_+(\sigma, \sigma')\end{aligned}\tag{5.55}$$

The algebra between the constraints (5.25) now gets modified to

$$\begin{aligned}\{\tilde{J}_1(\sigma), \tilde{J}_1(\sigma')\} &= -i(\chi_1(\sigma) - 2\chi_2(\sigma))\delta_P(\sigma - \sigma') , \\ \{\tilde{J}_2(\sigma), \tilde{J}_2(\sigma')\} &= -i(\chi_1(\sigma) + 2\chi_2(\sigma))\delta_P(\sigma - \sigma') , \\ \{\tilde{J}_1(\sigma), \tilde{J}_2(\sigma')\} &= -i(\chi_1(\sigma) - 2\chi_2(\sigma))\delta_P(\sigma + \sigma') .\end{aligned}\tag{5.56}$$

The algebra between $\tilde{J}(\sigma)$ and $\chi(\sigma)$ can now be computed by using the modified bracket (5.44) to get,

$$\begin{aligned}\{\chi_1(\sigma), \tilde{J}_1(\sigma')\} &= -(2\tilde{J}_1(\sigma) + \tilde{J}_1(\sigma'))\partial_\sigma\delta_P(\sigma - \sigma') + (2\tilde{J}_2(\sigma) + \tilde{J}_1(\sigma'))\partial_\sigma\delta_P(\sigma + \sigma') , \\ \{\chi_1(\sigma), \tilde{J}_2(\sigma')\} &= (2\tilde{J}_2(\sigma) + \tilde{J}_2(\sigma'))\partial_\sigma\delta_P(\sigma - \sigma') - (2\tilde{J}_1(\sigma) + \tilde{J}_2(\sigma'))\partial_\sigma\delta_P(\sigma + \sigma') , \\ \{\chi_2(\sigma), \tilde{J}_1(\sigma')\} &= \left(\tilde{J}_1(\sigma) + \frac{1}{2}\tilde{J}_1(\sigma')\right)\partial_\sigma\delta_P(\sigma - \sigma') + \left(\tilde{J}_2(\sigma) + \frac{1}{2}\tilde{J}_1(\sigma')\right)\partial_\sigma\delta_P(\sigma + \sigma') , \\ \{\chi_2(\sigma), \tilde{J}_2(\sigma')\} &= \left(\tilde{J}_2(\sigma) + \frac{1}{2}\tilde{J}_2(\sigma')\right)\partial_\sigma\delta_P(\sigma - \sigma') + \left(\tilde{J}_1(\sigma) + \frac{1}{2}\tilde{J}_2(\sigma')\right)\partial_\sigma\delta_P(\sigma + \sigma')\end{aligned}\tag{5.57}$$

⁵Since there is no translational symmetry, the presence of $\delta_P(\sigma + \sigma')$ does not give rise to any inconsistency here.

which clearly displays a new structure for the super-Virasoro algebra.

As a matter of consistency, we write down the hamiltonian of the superstring and then study the time evolution of the ψ_{\pm} modes. This follows easily from the Virasoro functional $L[f]$ (5.38) by setting $f(\sigma) = e^{im\sigma}$, which gives

$$L_m = \frac{1}{4} \int_{-\pi}^{\pi} d\sigma e^{-im\sigma} [\{\Pi(\sigma) + \partial_{\sigma} X(\sigma)\}^2 + 2i\psi_{+}^{\mu} \partial_{\sigma} \psi_{\mu+}] \quad (5.58)$$

Setting $m = 0$, gives the hamiltonian

$$\begin{aligned} H = L_0 &= \frac{1}{4} \int_{-\pi}^{\pi} d\sigma [\{\Pi(\sigma) + \partial_{\sigma} X(\sigma)\}^2 + 2i\psi_{+}^{\mu} \partial_{\sigma} \psi_{\mu+}] \\ &= \frac{1}{2} \int_0^{\pi} d\sigma [\Pi^2(\sigma) + \partial_{\sigma} X(\sigma)^2 + i(\psi_{+}^{\mu}(\sigma) \partial_{\sigma} \psi_{\mu+}(\sigma) - \psi_{-}^{\mu}(\sigma) \partial_{\sigma} \psi_{\mu-}(\sigma))] . \end{aligned} \quad (5.59)$$

This immediately leads to

$$\dot{\psi}_{-}(\sigma) = \{\psi_{-}(\sigma), H\} = -\partial_{\sigma} \psi_{-}(\sigma) \quad ; \quad \dot{\psi}_{+}(\sigma) = \{\psi_{+}(\sigma), H\} = \partial_{\sigma} \psi_{+}(\sigma) , \quad (5.60)$$

which are precisely the equations of motion for the fermionic fields. One can therefore regard (5.44) and (5.52) as the final symplectic structure of the free superstring theory.

5.4 The interacting theory :

The action for a super string moving in the presence of a constant background Neveu-Schwarz two form field $\mathcal{F}_{\mu\nu}$ is given by,

$$\begin{aligned} S &= -\frac{1}{2} \int_{\Sigma} d^2\sigma \left(\eta_{\mu\nu} \partial_a X^{\mu} \partial^a X^{\nu} + \epsilon^{ab} \mathcal{F}_{\mu\nu} \partial_a X^{\mu} \partial_b X^{\nu} \right. \\ &\quad \left. - i\bar{\psi}^{\mu} \rho^a \partial_a \psi_{\mu} + i\mathcal{F}_{\mu\nu} \bar{\psi}^{\mu} \rho_b \epsilon^{ab} \partial_a \psi^{\nu} \right) . \end{aligned} \quad (5.61)$$

The bosonic and fermionic sectors decouple. We consider just the fermionic sector since the bosonic sector has already been discussed in chapter 2. In component the fermionic sector reads

$$S_F = \frac{i}{2} \int_{\Sigma} d\tau d\sigma \left(\psi_{-}^{\mu} \partial_{+} \psi_{-\mu} + \psi_{+}^{\mu} \partial_{-} \psi_{+\mu} - \mathcal{F}_{\mu\nu} \psi_{-}^{\mu} \partial_{+} \psi_{-}^{\nu} + \mathcal{F}_{\mu\nu} \psi_{+}^{\mu} \partial_{-} \psi_{+}^{\nu} \right) . \quad (5.62)$$

The minimum action principle $\delta S = 0$ leads to a volume term that vanishes when the equations of motion hold, and also to a surface term

$$\left(\psi_-^\mu (\eta_{\mu\nu} - \mathcal{F}_{\mu\nu}) \delta \psi_-^\nu - \psi_+^\mu (\eta_{\mu\nu} + \mathcal{F}_{\mu\nu}) \delta \psi_+^\nu \right) \Big|_0^\pi = 0 . \quad (5.63)$$

It is not possible to find non trivial boundary conditions involving ψ_-^μ and ψ_+^μ that makes the above surface term vanish. However, the addition of a boundary term [57], [81]

$$S_{bound} = \frac{i}{2\pi\alpha'} \int_\Sigma d\tau d\sigma \left(\mathcal{F}_{\mu\nu} \psi_+^\mu \partial_- \psi_+^\nu \right) \quad (5.64)$$

makes it possible to find a solution to the boundary condition. Addition of this term to S_F leads to the total action:

$$S = \frac{-i}{4\pi\alpha'} \int_\Sigma d\tau d\sigma \left(\psi_-^\mu E_{\nu\mu} \partial_+ \psi_-^\nu + \psi_+^\mu E_{\nu\mu} \partial_- \psi_+^\nu \right) , \quad (5.65)$$

where $E^{\mu\nu} = \eta^{\mu\nu} + \mathcal{F}^{\mu\nu}$. The corresponding boundary term coming from $\delta S = 0$ is given by

$$\left(\psi_-^\mu E_{\nu\mu} \delta \psi_-^\nu - \psi_+^\mu E_{\nu\mu} \delta \psi_+^\nu \right) \Big|_0^\pi = 0. \quad (5.66)$$

The above condition is satisfied by the following conditions that preserve supersymmetry [82] at the string endpoints $\sigma = 0$ and $\sigma = \pi$:

$$\begin{aligned} E_{\nu\mu} \psi_+^\nu(0, \tau) &= E_{\mu\nu} \psi_-^\nu(0, \tau) , \\ E_{\nu\mu} \psi_+^\nu(\pi, \tau) &= \lambda E_{\mu\nu} \psi_-^\nu(\pi, \tau) , \end{aligned} \quad (5.67)$$

where $\lambda = \pm 1$ with the plus sign corresponding to Ramond boundary condition and the minus corresponding to the Neveu-Schwarz case. Here too we work with Ramond boundary conditions. Now the BCs are recast as

$$\left(E_{\nu\mu} \psi_{(+)}^\nu(\sigma, \tau) - E_{\mu\nu} \psi_{(-)}^\nu(\sigma, \tau) \right) \Big|_{\sigma=0,\pi} = 0 . \quad (5.68)$$

This nontrivial BC leads to a modification in the original (naive) (5.11) DBs. The $\{\psi_{(+)}^\mu(\sigma, \tau), \psi_{(+)}^\nu(\sigma', \tau)\}_{DB}$ is the same as that of the free string (5.11). We therefore make an ansatz

$$\{\psi_+^\mu(\sigma, \tau), \psi_-^\nu(\sigma', \tau)\}_{DB} = C^{\mu\nu} \delta_P(\sigma + \sigma') . \quad (5.69)$$

Taking brackets between the BCs (5.68) and $\psi_-^\gamma(\sigma')$ we get

$$E_{\nu\mu} C^{\nu\gamma} = -i E_{\mu\gamma}. \quad (5.70)$$

Solving this, we find

$$C^{\mu\nu} = -i \left[(1 - \mathcal{F}^2)^{-1} \right]^{\mu\rho} E_{\rho\gamma} E^{\gamma\nu}. \quad (5.71)$$

One can also take brackets between the BCs (5.68) and $\psi_+^\gamma(\sigma')$, which yields

$$C^{\nu\mu} = -i \left[(1 - \mathcal{F}^2)^{-1} \right]^{\mu\rho} E_{\gamma\rho} E^{\nu\gamma}. \quad (5.72)$$

Although the expressions (5.71) and (5.72) look different, they are actually the same as one can see easily by taking transpose of (5.72) and using the fact that, for any matrix M we have the commuting property for the product : $f(M)g(M) = g(M)f(M)$, holding for any two polynomials f and g of M . In other words, f and g can be regarded as functions which map matrices to matrices of same dimension and are constructed out of the same matrix M . Finally we can write the matrix $C = \{C^{\mu\nu}\}$ more compactly as

$$C = -i \left[(1 - \mathcal{F}^2)^{-1} (1 + \mathcal{F})^2 \right]. \quad (5.73)$$

We therefore get the following modification:

$$\{\psi_+^\mu(\sigma, \tau), \psi_-^\nu(\sigma', \tau)\}_{DB} = -i \left[(1 - \mathcal{F}^2)^{-1} \right]^{\mu\rho} E_{\rho\gamma} E^{\gamma\nu} \delta_P(\sigma + \sigma'), \quad (5.74)$$

which also reduces to those of [58], upto the $\delta_P(\sigma + \sigma')$ factor. Finally, note that in the limit $\mathcal{F}_{\mu\nu} \rightarrow 0$ (5.74), the last of (5.52) is reproduced.

5.5 Summary

In this chapter we have extended the methodology of chapter 2 and chapter 3 to an open fermionic string propagating freely and one moving in a constant antisymmetric background field. Here also we have observed that boundary conditions are incompatible with the basic brackets. So the modification of the canonical bracket was necessary. Eventually we have

constructed the appropriate delta function for the physical interval $[0, \pi]$ of the string and in the process we have obtained the non(anti)commutative structure of the super string. Finally the above non(anti)commutative structure led to new results in the algebra of superconstraints which still remain involutive, indicating the internal consistency of our analysis.

Chapter 6

Normal ordering and non(anti)commutativity in open super strings

In this chapter we study non(anti)commutativity in an open super string moving in the presence of a background antisymmetric tensor field $\mathcal{B}_{\mu\nu}$ in a conformal field theoretic approach.

In chapter 4, noncommutativity in an open bosonic string moving in the presence of a background Neveu-Schwarz two-form field $\mathcal{B}_{\mu\nu}$ is investigated in a conformal field theory approach. The mode algebra is first obtained using the newly proposed normal ordering, which satisfies both equations of motion and BC(s). Using these the commutator among the string coordinates is obtained. Interestingly, this new normal ordering yields the same algebra between the modes as the one satisfying only the equations of motion. In this approach, we find that noncommutativity originates more transparently and our results match with the existing results in the literature. In this chapter, we extend the same methodology to analyse an open super string propagating freely and one moving in a constant antisymmetric background field. To start with we discuss the recent results involving new normal ordered products (of fermionic operators) in [62]. Then we study the symplectic structure of the fermionic sector of both free and interacting super string. The computational details of some of the key results in the chapter are given in

the appendix B.

6.1 New Normal ordering for fermionic string coordinates

The action for a super string moving in the presence of a constant background antisymmetric tensor field $\mathcal{B}_{\mu\nu}$ is given by:

$$S = \frac{-1}{4\pi\alpha'} \int_{\Sigma} d\tau d\sigma \left[\partial_a X^\mu \partial^a X_\mu + \epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + i\psi_{\mu(-)} E^{\nu\mu} \partial_+ \psi_{\nu(-)} + i\psi_{\mu(+)} E^{\nu\mu} \partial_- \psi_{\nu(+)} \right] \quad (6.1)$$

where, $\partial_+ = \partial_\tau + \partial_\sigma$, $\partial_- = \partial_\tau - \partial_\sigma$ and $E^{\mu\nu} = \eta^{\mu\nu} + \mathcal{B}^{\mu\nu}$.

Now since the bosonic and fermionic sectors decouple, we can consider the fermionic sector separately¹. The variation of the fermionic part of the action (6.1) gives the classical equations of motion:

$$\partial_+ \psi_{\nu(-)} = 0 \quad , \quad \partial_- \psi_{\nu(+)} = 0 \quad (6.2)$$

and a boundary term that yields the following BCs:

$$\begin{aligned} E_{\nu\mu} \psi_{(+)}^\nu(0, \tau) &= E_{\mu\nu} \psi_{(-)}^\nu(0, \tau) \\ E_{\nu\mu} \psi_{(+)}^\nu(\pi, \tau) &= \lambda E_{\mu\nu} \psi_{(-)}^\nu(\pi, \tau) \end{aligned} \quad (6.3)$$

at the endpoints $\sigma = 0$ and $\sigma = \pi$ of the string, where $\lambda = \pm 1$ corresponds to Ramond and Neveu-Schwarz BC(s) respectively.

It is convenient now to change to complex world-sheet coordinates and therefore we first make a Wick rotation by defining $\sigma^2 = i\tau$. Then we introduce the complex world sheet coordinates [53]: $z = \sigma^1 + i\sigma^2$; $\bar{z} = \sigma^1 - i\sigma^2$ and $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$. In this notation the fermionic part of the action (6.1) reads:

$$S_F = \frac{-i}{4\pi\alpha'} \int_{\Sigma} dz d\bar{z} [\psi_{\mu(-)} E^{\nu\mu} \partial_{\bar{z}} \psi_{\nu(-)} + \psi_{\mu(+)} E^{\nu\mu} \partial_z \psi_{\nu(+)}] \quad (6.4)$$

¹The bosonic sector was already discussed in chapter 4.

while the classical equations of motion (6.2) and the Ramond BCs (6.3) take the form:

$$\partial_{\bar{z}}\psi_{\nu(-)} = 0 \ , \ \partial_z\psi_{\nu(+)} = 0 \quad (6.5)$$

$$\left(E_{\nu\mu}\psi_{(+)}^{\nu}(z,\bar{z}) - E_{\mu\nu}\psi_{(-)}^{\nu}(z,\bar{z})\right)|_{z=-\bar{z}, 2\pi-\bar{z}} = 0. \quad (6.6)$$

We now study the properties of quantum operators corresponding to the classical variables by considering the expectation values [53]. Using the fact that the path integral of a total functional derivative vanishes and considering the insertion of one fermionic operator one finds:

$$\int [d\psi] \left[\frac{\delta}{\delta\psi_{(a)}^{\mu}(z,\bar{z})} [e^{-S_F}\psi_{(b)}^{\nu}(z',\bar{z}')] \right] = 0 \quad (6.7)$$

where, $a, b = +, -$. Considering first the case of $\psi_{(b)}^{\nu}(z',\bar{z}')$ inside the world-sheet and not at the boundary, this equation yields the following expectation values:

$$\begin{aligned} \langle \partial_z\psi_{(+)}^{\mu}(z,\bar{z})\psi_{(+)}^{\nu}(z',\bar{z}') \rangle &= 2\pi i \alpha' \langle \eta^{\mu\nu} \delta^2(z-z', \bar{z}-\bar{z}') \rangle \\ \langle \partial_{\bar{z}}\psi_{(-)}^{\mu}(z,\bar{z})\psi_{(-)}^{\nu}(z',\bar{z}') \rangle &= 2\pi i \alpha' \langle \eta^{\mu\nu} \delta^2(z-z', \bar{z}-\bar{z}') \rangle \\ \langle \partial_{\bar{z}}\psi_{(-)}^{\mu}(z,\bar{z})\psi_{(+)}^{\nu}(z',\bar{z}') \rangle &= \langle \partial_z\psi_{(+)}^{\mu}(z,\bar{z})\psi_{(-)}^{\nu}(z',\bar{z}') \rangle = 0. \end{aligned} \quad (6.8)$$

Using these results one finds the appropriate way to define normal ordered products that satisfy the equations of motion for fermionic operators that are not at the world-sheet boundary [62, 83]:

$$\begin{aligned} : \psi_{(+)}^{\mu}(z,\bar{z}) \psi_{(+)}^{\nu}(z',\bar{z}') : &= \psi_{(+)}^{\mu}(z,\bar{z}) \psi_{(+)}^{\nu}(z',\bar{z}') - \frac{i\alpha'}{\bar{z}-\bar{z}'} \eta^{\mu\nu} \\ : \psi_{(-)}^{\mu}(z,\bar{z}) \psi_{(-)}^{\nu}(z',\bar{z}') : &= \psi_{(-)}^{\mu}(z,\bar{z}) \psi_{(-)}^{\nu}(z',\bar{z}') - \frac{i\alpha'}{z-z'} \eta^{\mu\nu} \\ : \psi_{(+)}^{\mu}(z,\bar{z}) \psi_{(-)}^{\nu}(z',\bar{z}') : &= 0 \\ : \psi_{(-)}^{\mu}(z,\bar{z}) \psi_{(+)}^{\nu}(z',\bar{z}') : &= 0. \end{aligned} \quad (6.9)$$

The above products satisfy the equations of motion (6.5) at the quantum level, but fails to satisfy the BC(s) (6.6).

At this point it is more convenient to choose world sheet coordinates, related to these z coordinates by conformal transformation, that simplify the representation of the boundary,

$$\omega = \exp(-iz) = e^{-i\sigma^1+\sigma^2} \ ; \ \bar{\omega} = e^{i\sigma^1+\sigma^2}. \quad (6.10)$$

Besides replacing $\exp(-iz) \rightarrow \omega$, we must transform the fields [83],

$$\psi_{\omega^{\frac{1}{2}}}^{\mu}(\omega) = (\partial_{\omega} z)^{\frac{1}{2}} \psi_{z^{\frac{1}{2}}}^{\mu}(z) = i^{\frac{1}{2}} \omega^{-\frac{1}{2}} \psi_{z^{\frac{1}{2}}}^{\mu}(z). \quad (6.11)$$

The subscripts are a reminder that these transform with half the weight of a vector. In this present coordinates the complete boundary corresponds just to the region $\omega = \bar{\omega}$. Further, the action (6.4) along with equations of motion (6.5) in terms of $\omega, \bar{\omega}$ has still the same form, while the form of BC(s) (6.6) change to the following:

$$\left(E_{\nu\mu} \psi_{(+)}^{\nu}(\omega, \bar{\omega}) + i E_{\mu\nu} \psi_{(-)}^{\nu}(\omega, \bar{\omega}) \right) |_{\omega=\bar{\omega}} = 0. \quad (6.12)$$

Let us now consider the case of an insertion of a fermionic string coordinate $\psi_{(\pm)}^{\nu}(\omega')$ located at the world-sheet boundary. Note that since $\omega' = \bar{\omega}'$ at the boundary, the fermionic coordinate insertion at the boundary depends only on ω' . Working out equation (6.7), but now subject to constraint (6.12) (with ω replaced by ω' in (6.12)), we find² (see appendix B for the computational details):

$$\begin{aligned} \langle \partial_{\omega} \psi_{(+)}^{\mu}(\omega, \bar{\omega}) \psi_{(+)}^{\nu}(\omega') \rangle &= 2\pi i \alpha' \langle \eta^{\mu\nu} \delta^2(\omega - \omega', \bar{\omega} - \omega') \rangle \\ \langle \partial_{\omega} \psi_{(-)}^{\mu}(\omega, \bar{\omega}) \psi_{(-)}^{\nu}(\omega') \rangle &= 2\pi i \alpha' \langle \eta^{\mu\nu} \delta^2(\omega - \omega', \bar{\omega} - \omega') \rangle \\ \langle \partial_{\omega} \psi_{(+)}^{\mu}(\omega, \bar{\omega}) \psi_{(-)}^{\nu}(\omega') \rangle &= -2\pi i \alpha' \langle [(\eta + \mathcal{B})^{-1} (\eta - \mathcal{B})]^{\nu\mu} \delta^2(\omega - \omega', \bar{\omega} - \omega') \rangle \\ \langle \partial_{\omega} \psi_{(-)}^{\mu}(\omega, \bar{\omega}) \psi_{(+)}^{\nu}(\omega') \rangle &= 2\pi i \alpha' \langle [(\eta + \mathcal{B})^{-1} (\eta - \mathcal{B})]^{\nu\mu} \delta^2(\omega - \omega', \bar{\omega} - \omega') \rangle. \end{aligned} \quad (6.13)$$

So the appropriate normal ordering for fermionic string coordinates at the boundary reads:

$$\begin{aligned} : \psi_{(+)}^{\mu}(\omega, \bar{\omega}) \psi_{(+)}^{\nu}(\omega') : &= \psi_{(+)}^{\mu}(\omega, \bar{\omega}) \psi_{(+)}^{\nu}(\omega') - \frac{i\alpha'}{(\bar{\omega} - \omega')} \eta^{\mu\nu} \\ : \psi_{(-)}^{\mu}(\omega, \bar{\omega}) \psi_{(-)}^{\nu}(\omega') : &= \psi_{(-)}^{\mu}(\omega, \bar{\omega}) \psi_{(-)}^{\nu}(\omega') - \frac{i\alpha'}{(\omega - \omega')} \eta^{\mu\nu} \\ : \psi_{(+)}^{\mu}(\omega, \bar{\omega}) \psi_{(-)}^{\nu}(\omega') : &= \psi_{(+)}^{\mu}(\omega, \bar{\omega}) \psi_{(-)}^{\nu}(\omega') + \frac{\alpha' [(\eta + \mathcal{B})^{-1} (\eta - \mathcal{B})]^{\nu\mu}}{(\bar{\omega} - \omega')} \\ : \psi_{(-)}^{\mu}(\omega, \bar{\omega}) \psi_{(+)}^{\nu}(\omega') : &= \psi_{(-)}^{\mu}(\omega, \bar{\omega}) \psi_{(+)}^{\nu}(\omega') - \frac{\alpha' [(\eta + \mathcal{B})^{-1} (\eta - \mathcal{B})]^{\nu\mu}}{(\omega - \omega')} \end{aligned} \quad (6.14)$$

²Note that the fields ψ 's with unprimed arguments are not located at the boundary.

The above results of normal ordering of fermionic operators are new and incorporates the effect of BC(s).

Now for the functional $\mathcal{F}[X]$ (representing the combinations occurring in the left hand side of the above equation), the new normal ordering (in absence of the \mathcal{B} field) can be compactly written as:

$$: \mathcal{F} := \exp \left(\frac{i\alpha'}{2} \int d^2\omega'' d^2\omega''' \left[\frac{1}{(\omega'' - \omega''')} \frac{\delta}{\delta\psi_{(-)}^\mu(\omega'', \bar{\omega}'')} \frac{\delta}{\delta\psi_{\mu(-)}(\omega''', \bar{\omega}''')} + (\omega \leftrightarrow \bar{\omega}, - \leftrightarrow +) \right] \right) \mathcal{F} \quad (6.15)$$

Note that the fields ψ 's with double prime and triple prime arguments in (6.15) are not located at the boundary.

We shall see now that normal ordered products are important to compute the central charge which gives us the critical dimension. The energy-momentum tensor (in the absence of the \mathcal{B} field) for the fermionic sector for points inside the world-sheet (in the z -frame) is given by:

$$\begin{aligned} T^{zz} &= -\frac{1}{2} \psi_{\mu(+)} \partial_{\bar{z}} \psi_{(+)}^\mu \equiv \bar{T} \\ T^{\bar{z}\bar{z}} &= -\frac{1}{2} \psi_{\mu(-)} \partial_z \psi_{(-)}^\mu \equiv T \end{aligned} \quad (6.16)$$

while at the boundary, the BC(s) (6.6) (with $\mathcal{B} = 0$) relating $\psi_{\nu(-)}$ to $\psi_{\nu(+)}$ lead to :

$$\bar{T} = -\frac{1}{2} \psi_{\mu(+)} \partial_{\bar{z}} \psi_{(+)}^\mu = -T \quad (6.17)$$

where we have used $\partial_{\bar{z}} = -\partial_z$ (since $dz = -d\bar{z}$ at the boundary). The central charge can now be computed from the most singular term in the normal ordered product of energy-momentum tensor. This involves two contractions of the fermionic coordinate operator products and is proportional to [62]:

$$\begin{aligned} & \int dz' \dots dz''' \frac{1}{2} \left[\frac{i\alpha'}{(z' - z'')} \frac{\delta^2}{\delta\psi_{\mu(-)}(z') \delta\psi_{(-)}^\mu(z'')} \right] \left[\frac{i\alpha'}{(z''' - z''')} \frac{\delta^2}{\delta\psi_{\mu(-)}(z''') \delta\psi_{(-)}^\mu(z''')} \right] \\ & \times [T(z_1)T(z_2)] \\ & \sim \frac{D\alpha'^2}{4(z_1 - z_2)^4} \end{aligned} \quad (6.18)$$

where \sim mean “equal up to nonsingular terms”³. The above computation gives the well known result $D/2$ as the central charge where D is the dimension of space-time [70], [83]. The results are also in conformity with [62].

We shall make use of the results discussed here in the next section where we study both free and interacting open super strings.

6.2 Mode expansions and Non(anti)Commutativity for super strings

6.2.1 Free open strings

In this section, we consider the mode expansions of free ($\mathcal{B}_{\mu\nu} = 0$) open super strings. We first expand $\psi_{(-)}^{\mu}(z)$ and $\psi_{(+)}^{\mu}(\bar{z})$ in Fourier modes in (z, \bar{z}) coordinates [83]:

$$\psi_{(-)}^{\mu}(z) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbf{Z}} d_m^{\mu} \exp(imz) \quad ; \quad \psi_{(+)}^{\mu}(\bar{z}) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbf{Z}} \tilde{d}_m^{\mu} \exp(-im\bar{z}). \quad (6.19)$$

Let us also write these as Laurent expansions in $(\omega, \bar{\omega})$ coordinates:

$$\psi_{(-)}^{\mu}(\omega) = \frac{i^{\frac{1}{2}}}{\sqrt{2\pi}} \sum_{m \in \mathbf{Z}} \frac{d_m^{\mu}}{\omega^{m+\frac{1}{2}}} \quad ; \quad \psi_{(+)}^{\mu}(\bar{\omega}) = \frac{i^{-\frac{1}{2}}}{\sqrt{2\pi}} \sum_{m \in \mathbf{Z}} \frac{\tilde{d}_m^{\mu}}{\bar{\omega}^{m+\frac{1}{2}}}. \quad (6.20)$$

Now the BC(s) (6.12) in case of free open super strings ($\mathcal{B}_{\mu\nu} = 0$) requires $d = \tilde{d}$ in the expansions (6.20). The expressions (6.20) can be equivalently written as:

$$d_m^{\mu} = \frac{\sqrt{2\pi}}{\sqrt{i}} \oint \frac{d\omega}{2\pi i} \omega^{m-\frac{1}{2}} \psi_{(-)}^{\mu}(\omega) = -\sqrt{2\pi}\sqrt{i} \oint \frac{d\bar{\omega}}{2\pi i} \bar{\omega}^{m-\frac{1}{2}} \psi_{(+)}^{\mu}(\bar{\omega}). \quad (6.21)$$

The anticommutation relation between d 's can be worked out from the contour argument [53] and the operator product expansion (OPE) (6.14) (with $\mathcal{B}_{\mu\nu} = 0$):

$$\begin{aligned} \{d_m^{\mu}, d_n^{\nu}\} &= \frac{1}{i} \oint \frac{d\omega_2}{2\pi i} \text{Res}_{\omega_1 \rightarrow \omega_2} \left(\omega_1^{m-\frac{1}{2}} \psi_{(-)}^{\mu}(\omega_1) \omega_2^{n-\frac{1}{2}} \psi_{(-)}^{\nu}(\omega_2) \right) \\ &= 2\pi\alpha' \eta^{\mu\nu} \delta_{m+n,0} = \eta^{\mu\nu} \delta_{m+n,0} \end{aligned} \quad (6.22)$$

³The other less singular terms are not given explicitly.

where we have set $2\pi\alpha' = 1$. The anti-commutation relations between $\psi_{(-)}^{\mu}(\omega, \bar{\omega})$ and $\psi_{(+)}^{\nu}(\omega', \bar{\omega}')$ are then obtained by using (6.22):

$$\begin{aligned}\left\{\psi_{(-)}^{\mu}(\omega, \bar{\omega}), \psi_{(-)}^{\nu}(\omega', \bar{\omega}')\right\} &= \frac{i\eta^{\mu\nu}}{2\pi} \sum_{m \in \mathbf{Z}} \left(\omega^{-m-\frac{1}{2}} \omega'^{m-\frac{1}{2}}\right) \\ \left\{\psi_{(+)}^{\mu}(\omega, \bar{\omega}), \psi_{(+)}^{\nu}(\omega', \bar{\omega}')\right\} &= -\frac{i\eta^{\mu\nu}}{2\pi} \sum_{m \in \mathbf{Z}} \left(\bar{\omega}^{-m-\frac{1}{2}} \bar{\omega}'^{m-\frac{1}{2}}\right) \\ \left\{\psi_{(-)}^{\mu}(\omega, \bar{\omega}), \psi_{(+)}^{\nu}(\omega', \bar{\omega}')\right\} &= \frac{\eta^{\mu\nu}}{2\pi} \sum_{m \in \mathbf{Z}} \left(\omega^{-m-\frac{1}{2}} \bar{\omega}'^{m-\frac{1}{2}}\right).\end{aligned}\quad (6.23)$$

To obtain the usual equal time ($\tau = \tau'$) anticommutation relation we first rewrite (6.23) in “ z frame” using (6.10, 6.11) and then in terms of σ^1, σ^2 to find:

$$\begin{aligned}\left\{\psi_{(-)}^{\mu}(\sigma^1, \sigma^2), \psi_{(-)}^{\nu}(\sigma'^1, \sigma'^2)\right\} &= \frac{\eta^{\mu\nu}}{2\pi} \sum_{m \in \mathbf{Z}} \left[\exp\left(im(\sigma^1 + i\sigma^2 - \sigma'^1 - i\sigma'^2)\right)\right] \\ \left\{\psi_{(+)}^{\mu}(\sigma^1, \sigma^2), \psi_{(+)}^{\nu}(\sigma'^1, \sigma'^2)\right\} &= \frac{\eta^{\mu\nu}}{2\pi} \sum_{m \in \mathbf{Z}} \left[\exp\left(im(\sigma^1 - i\sigma^2 - \sigma'^1 + i\sigma'^2)\right)\right] \\ \left\{\psi_{(-)}^{\mu}(\sigma^1, \sigma^2), \psi_{(+)}^{\nu}(\sigma'^1, \sigma'^2)\right\} &= \frac{\eta^{\mu\nu}}{2\pi} \sum_{m \in \mathbf{Z}} \left[\exp\left(im(\sigma^1 + i\sigma^2 + \sigma'^1 - i\sigma'^2)\right)\right].\end{aligned}\quad (6.24)$$

Finally substituting $\tau = \tau'$ (i.e. $\sigma^2 = \sigma'^2$) and $\sigma^1 = \sigma$ we get back the equal time anti-commutation relations:

$$\begin{aligned}\left\{\psi_{(-)}^{\mu}(\sigma, \tau), \psi_{(-)}^{\nu}(\sigma', \tau)\right\} &= \eta^{\mu\nu} \delta_P(\sigma - \sigma') \\ \left\{\psi_{(+)}^{\mu}(\sigma, \tau), \psi_{(+)}^{\nu}(\sigma', \tau)\right\} &= \eta^{\mu\nu} \delta_P(\sigma - \sigma') \\ \left\{\psi_{(-)}^{\mu}(\sigma, \tau), \psi_{(+)}^{\nu}(\sigma', \tau)\right\} &= \eta^{\mu\nu} \delta_P(\sigma + \sigma').\end{aligned}\quad (6.25)$$

where, $\delta_P(\sigma - \sigma')$ is the so called periodic delta function which is defined as:

$$\delta_P(\sigma - \sigma') = \frac{1}{2\pi} \sum_{m \in \mathbf{Z}} \exp(im(\sigma - \sigma')). \quad (6.26)$$

This structure of anticommutator is completely consistent with the BCs (6.12) for $\mathcal{B}_{\mu\nu} = 0$. Note that not only the usual Dirac delta function is replaced by the periodic delta function but also the anticommutator among $\psi_{(-)}, \psi_{(+)}$ are non-vanishing even in case of the free open fermionic string as we have already seen in previous chapter [59, 60]. The most important feature of the above analysis is that unlike the bosonic case, the new normal ordering of the fermionic operators (that incorporates the BC(s)) (6.14) leads to the nonanticommutative structures (6.25) among the fermionic string coordinates.

6.2.2 Open superstring in the constant \mathcal{B} -field background

We now analyse the open superstring moving in presence of a background antisymmetric tensor field $\mathcal{B}_{\mu\nu}$. To begin with, let us again consider the Laurent expansion of $\psi_{(-)}^\mu(\omega)$ and $\psi_{(+)}^\mu(\bar{\omega})$ (6.20). Now due to the BC(s) (6.12) (with $\mathcal{B}_{\mu\nu} \neq 0$), the modes d and \tilde{d} are no longer independent but satisfy the following relation:

$$E_{\mu\nu} d_m^\nu = E_{\nu\mu} \tilde{d}_m^\nu. \quad (6.27)$$

Hence there exists only one set of independent modes α_m^μ , which can be thought of as the modes of free open strings and is related to d_m^μ and \tilde{d}_m^μ by:

$$\begin{aligned} d_m^\mu &= (\delta^\mu_\nu - \mathcal{B}^\mu_\nu) \alpha_m^\nu := [(\mathbb{1} - \mathcal{B})\alpha]_m^\mu \\ \tilde{d}_m^\mu &= (\delta^\mu_\nu + \mathcal{B}^\mu_\nu) \alpha_m^\nu := [(\mathbb{1} + \mathcal{B})\alpha]_m^\mu. \end{aligned} \quad (6.28)$$

Note that under world-sheet parity transformation (i.e. $\sigma \leftrightarrow -\sigma$), $d_m^\mu \leftrightarrow \tilde{d}_m^\mu$, since $\mathcal{B}_{\mu\nu}$ is a world-sheet pseudo-scalar (similar to bosonic part). Substituting (6.28) in (6.20), we obtain the following Laurent expansions for ψ_-^μ and ψ_+^μ :

$$\begin{aligned} \psi_{(-)}^\mu(\omega) &= \frac{i^{\frac{1}{2}}}{\sqrt{2\pi}} \sum_{m \in \mathbf{Z}} \frac{[(\mathbb{1} - \mathcal{B})\alpha]_m^\mu}{\omega^{m+\frac{1}{2}}} \\ \psi_{(+)}^\mu(\bar{\omega}) &= \frac{i^{-\frac{1}{2}}}{\sqrt{2\pi}} \sum_{m \in \mathbf{Z}} \frac{[(\mathbb{1} + \mathcal{B})\alpha]_m^\mu}{\bar{\omega}^{m+\frac{1}{2}}}. \end{aligned} \quad (6.29)$$

These are the appropriate mode expansions for the fermionic part of the interacting superstring, that satisfy both the equations of motion (6.5) and the BC(s) (6.12).

Now the expressions (6.29) for interacting superstrings can also be written as:

$$\begin{aligned} [(\mathbb{1} - \mathcal{B})\alpha]_m^\mu &= \frac{\sqrt{2\pi}}{i} \oint \frac{d\omega}{2\pi i} \omega^{m-\frac{1}{2}} \psi_{(-)}^\mu(\omega) \\ [(\mathbb{1} + \mathcal{B})\alpha]_m^\mu &= \frac{\sqrt{2\pi}}{i} \oint \frac{d\bar{\omega}}{2\pi i} \bar{\omega}^{m-\frac{1}{2}} \psi_{(+)}^\mu(\bar{\omega}). \end{aligned} \quad (6.30)$$

The anticommutation relation between α 's can be obtained once again from the contour argument (using (6.30)) and the $\psi\psi$ OPE (6.14):

$$\{\alpha_m^\mu, \alpha_n^\nu\} = \left[(\mathbb{1} - \mathcal{B}^2)^{-1} \right]^{\mu\nu} \delta_{m,-n} = (\mathcal{M}^{-1})^{\mu\nu} \delta_{m,-n} \quad (6.31)$$

where, $\mathcal{M} = (\mathbb{1} - \mathcal{B}^2)$; $(\mathcal{B}^2)^{\mu\nu} = \mathcal{B}^\mu{}_\rho \mathcal{B}^{\rho\nu}$ ⁴. Now the anticommutator between the fermionic string coordinates can be computed using (6.29), (6.31). The antibrackets between $\{\psi_{(-)}^\mu, \psi_{(-)}^\nu\}$ and $\{\psi_{(+)}^\mu, \psi_{(+)}^\nu\}$ are the same as that of free case but the anticommutator between $\psi_{(-)}^\mu$ and $\psi_{(+)}^\mu$ gets modified to the following form:

$$\{\psi_{(-)}^\mu(\omega, \bar{\omega}), \psi_{(+)}^\nu(\omega', \bar{\omega}')\} = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \left[\frac{(\mathbb{1} - \mathcal{B})^\mu{}_\rho \left[(\mathbb{1} - \mathcal{B}^2)^{-1} \right]^{\rho\sigma} (\mathbb{1} - \mathcal{B})_\sigma{}^\nu}{\omega^{m+\frac{1}{2}} \bar{\omega}'^{-m+\frac{1}{2}}} \right]. \quad (6.32)$$

Now proceeding as before, we can write the above anticommutation relation in (τ, σ) coordinates to obtain the usual equal time (i.e. $\tau = \tau'$) anticommutation relation:

$$\{\psi_{(-)}^\mu(\sigma, \tau), \psi_{(+)}^\nu(\sigma', \tau)\} = E^{\rho\mu} \left[(\mathbb{1} - \mathcal{B}^2)^{-1} \right]_{\rho\sigma} E^{\nu\sigma} \delta_P(\sigma + \sigma'). \quad (6.33)$$

The above result reduces to the the free case result in the $\mathcal{B}_{\mu\nu} = 0$ limit and also agrees with the existing results in the previous chapter and literature [59].

6.3 Summary

In this chapter, we have used conformal field theoretic techniques to compute the anticommutator among Fourier components of fermionic sector of super strings. Using this the anticommutator between the basic fermionic fields is obtained. This is the extension of our earlier work on bosonic strings discussed in chapter 4. The method is also different from ([59]), where the algebra among the Fourier components have been computed using the Faddeev-Jackiw symplectic formalism. The advantage of this approach is that the results one obtains takes into account the quantum effects right from the beginning, in contrary to the previous investigations, which were made essentially at the classical level [7, 8, 10, 39, 59]. Interestingly, the new normal ordering that takes into account the effect of the BC(s) plays a crucial role in obtaining the nonanticommutative symplectic structure among the fermionic string coordinates (6.25). This is in contrast to the analysis in case of the bosonic strings where the new normal ordering has no bearing on the symplectic structure. Finally, we also computed the oscillator algebra in

⁴Here we should note that $(\mathbb{1})^{\mu\nu} = \eta^{\mu\nu}$.

presence of the \mathcal{B} field which is a parity-odd field on the string world-sheet. As in the bosonic case, in presence of this \mathcal{B} field, the Fourier modes appearing in the Laurent series expansions of the fermionic fields $\psi_{(-)}^\mu$ and $\psi_{(+)}^\mu$ of the closed string are no longer equal when open string BCs are imposed. These rather get related to the free oscillator modes d_m^μ . Using these expressions of the modes, we rewrite the fermionic fields $\psi_{(-)}^\mu$ and $\psi_{(+)}^\mu$ entirely in terms of the free oscillator modes α_m^μ (6.28). Then a straight forward calculation, involving $\psi\psi$ OPE and contour argument yields the NC anticommutator, thereby reproducing the results of previous chapter.

Chapter 7

String non(anti)commutativity for Neveu-Schwarz boundary conditions

We have already seen in the case of fermionic string there is a choice between Ramond boundary conditions and Neveu Schwarz (NS) boundary conditions. Surprisingly a common point of all the studies in the superstring theory is that all the literatures are solely confined for the Ramond (R) BC(s) only and the second type of BC is less studied in the research area. Here in this chapter, we extend our methodology (which has already been discussed in chapter 5) to the superstring satisfying the NS boundary conditions. A nontrivial result we have found from the whole analysis is that, contrary to the R case, bosonic sector of the superstring satisfies Dirichlet BC at one end and Neumann BC at the other end provided the bosonic variable X^μ is allowed to be antiperiodic. This observation is completely new and has not been discussed elsewhere. Further, the symplectic structure of the bosonic sector also keeps the superconstraint algebra involutive. The bracket structures have also been computed using the mode expansions of the bosonic and the fermionic coordinates.

In the next section, the R-Neveu Schwarz (RNS) superstring action in the conformal gauge is briefly discussed to fix the notations. The section is then subdivided into two parts. In the first subsection, the BC(s) and the mode expansions of the fermionic sector of the superstring is given and the nonanticommutativity of the theory is revealed in the conventional Hamiltonian

framework. In the next subsection, the PB structure and the BC(s) of the bosonic sector is discussed. Then in next section we compute the super constraint algebra with the modified symplectic structure obtained in the previous section. The results obtained in the subsections 7.1.1 and 7.1.2 are further confirmed in section 7.3 by the mode expansion method. This consistency check is performed separately for the bosonic and the fermionic sector. Finally in Section 7.4 we discuss the non(anti)commutativity in the interacting superstring theory in the RNS formulation.

7.1 RNS free superstring

In the first part of this section we briefly mention the canonical algebra of the basic fields of a free open superstring (we have already discussed the superstring in chapter 5). Later we shall show how these algebraic structures get modified as a result of the boundary conditions of the theory. The action we take for our analysis is given by [79, 80]¹

$$S = -\frac{1}{2} \int_{\Sigma} d^2\sigma \left(\eta_{\mu\nu} \partial_a X^\mu \partial^a X^\nu - i \bar{\psi}^\mu \rho^a \partial_a \psi_\mu \right). \quad (7.1)$$

The bosonic and the fermionic part of the above action can be separated out as

$$S = S_B + S_F \quad (7.2)$$

where,

$$S_B = -\frac{1}{2} \int_{\Sigma} d^2\sigma \eta_{\mu\nu} \partial_a X^\mu \partial^a X^\nu \quad \text{and} \quad S_F = \frac{1}{2} \int_{\Sigma} d^2\sigma i \bar{\psi}^\mu \rho^a \partial_a \psi_\mu. \quad (7.3)$$

The components of the Majorana spinor ψ are denoted as ψ_{\pm}

$$\psi^\mu = \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix}. \quad (7.4)$$

¹We follow the same conventions as in chapter 5, $\rho^0 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\rho^1 = i\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and take the induced world-sheet metric and target space-time metric as $\eta^{ab} = \{-, +\}$, $\eta^{\mu\nu} = \{-, +, +, \dots, +\}$ respectively.

The Dirac antibracket of the first order action S_F is easily read off

$$\begin{aligned}\{\psi_+^\mu(\sigma), \psi_+^\nu(\sigma')\}_{D.B} &= \{\psi_-^\mu(\sigma), \psi_-^\nu(\sigma')\}_{D.B} = -i\eta^{\mu\nu}\delta(\sigma - \sigma') \\ \{\psi_+^\mu(\sigma), \psi_-^\nu(\sigma')\}_{D.B} &= 0.\end{aligned}\tag{7.5}$$

On the other hand the action S_B gives the following brackets among the bosonic variables

$$\begin{aligned}\{X^\mu(\sigma), \Pi^\nu(\sigma')\} &= \eta^{\mu\nu}\delta(\sigma - \sigma') \\ \{X^\mu(\sigma), X^\nu(\sigma')\} &= 0 = \{\Pi^\mu(\sigma), \Pi^\nu(\sigma')\}\end{aligned}\tag{7.6}$$

where Π_μ is the canonically conjugate momentum to X^μ , defined in the usual way. Eqs. (7.5) and (7.6) defines the preliminary symplectic structure of the theory. We shall now discuss the effects of BC(s) on these symplectic algebra for the fermionic and the bosonic sectors separately.

7.1.1 Fermionic sector

Varying the fermionic part of the action (7.3)

$$\delta S_F = i \int_{\Sigma} d^2\sigma \left[\delta\bar{\psi}_\mu \rho^a \partial_a \psi^\mu - \partial_\sigma (\psi_-^\mu \delta\psi_{\mu-} - \psi_+^\mu \delta\psi_{\mu+}) \right]\tag{7.7}$$

we obtain the Euler-Lagrange equation for the fermionic field

$$i\rho^a \partial_a \psi^\mu = 0.\tag{7.8}$$

together with the following BC(s):

$$\begin{aligned}\psi_+^\mu(0, \tau) &= \psi_-^\mu(0, \tau) \\ \psi_+^\mu(\pi, \tau) &= \lambda \psi_-^\mu(\pi, \tau)\end{aligned}\tag{7.9}$$

where $\lambda = \pm 1$ corresponds to the R BC(s) and the NS BC(s), respectively. In this chapter, we shall work with the NS BC(s) which we write in the following manner

$$(\psi_+^\mu(\sigma, \tau) - \psi_-^\mu(\sigma, \tau))|_{\sigma=0} = 0\tag{7.10}$$

$$(\psi_+^\mu(\sigma, \tau) + \psi_-^\mu(\sigma, \tau))|_{\sigma=\pi} = 0.\tag{7.11}$$

Now the mode expansion of the components of Majorana fermion, satisfying the above set of BC(s) is given by [79, 80]:

$$\begin{aligned}\psi_-^\mu(\sigma, \tau) &= \frac{1}{\sqrt{2}} \sum_{n \in Z + \frac{1}{2}} d_n^\mu e^{-in(\tau - \sigma)} \\ \psi_+^\mu(\sigma, \tau) &= \frac{1}{\sqrt{2}} \sum_{n \in Z + \frac{1}{2}} d_n^\mu e^{-in(\tau + \sigma)}.\end{aligned}\tag{7.12}$$

From the above mode expansions it follows automatically that

$$\psi_-^\mu(-\sigma, \tau) = \psi_+^\mu(\sigma, \tau).\tag{7.13}$$

Furthermore, making use of eq. (7.9), we obtain

$$\begin{aligned}\psi_\pm^\mu(\sigma = -\pi, \tau) &= -\psi_\pm^\mu(\sigma = \pi, \tau) \\ \psi_\pm^\mu(\sigma = -2\pi, \tau) &= \psi_\pm^\mu(\sigma = 2\pi, \tau)\end{aligned}\tag{7.14}$$

in the NS-sector. Hence $\psi_\pm^\mu(\sigma, \tau)$ is an antiperiodic function of antiperiodicity 2π which naturally implies that it is a periodic function of periodicity 4π . We now essentially follow the methodology discussed in chapter 5 for the present case. First, we introduce the antiperiodic delta function $\delta_{(a)P}(x)$ of antiperiodicity 2π and periodicity 4π

$$\delta_{(a)P}(x) = -\delta_{(a)P}(x + 2\pi) = \frac{1}{4\pi} \sum_{n \in Z + \frac{1}{2}} e^{inx}\tag{7.15}$$

which satisfies the defining property of a periodic δ -function i.e.

$$\int_{-2\pi}^{2\pi} dx' \delta_{(a)P}(x' - x) f(x') = f(x)\tag{7.16}$$

where $f(x)$ is an arbitrary periodic function with periodicity 4π . Using this we write the following expression for ψ_-^μ and ψ_+^μ in the physical interval $[0, \pi]$ of the string

$$2 \int_0^\pi d\sigma' [\delta_{(a)P}(\sigma' + \sigma) \psi_+^\mu(\sigma') + \delta_{(a)P}(\sigma' - \sigma) \psi_-^\mu(\sigma')] = \psi_-^\mu(\sigma)\tag{7.17}$$

$$2 \int_0^\pi d\sigma' [\delta_{(a)P}(\sigma' + \sigma) \psi_-^\mu(\sigma') + \delta_{(a)P}(\sigma' - \sigma) \psi_+^\mu(\sigma')] = \psi_+^\mu(\sigma).\tag{7.18}$$

We define a matrix $\Lambda_{AB}(\sigma, \sigma')$

$$\Lambda_{AB}(\sigma, \sigma') = \begin{pmatrix} \delta_{(a)P}(\sigma' - \sigma) & \delta_{(a)P}(\sigma' + \sigma) \\ \delta_{(a)P}(\sigma' + \sigma) & \delta_{(a)P}(\sigma' - \sigma) \end{pmatrix}\tag{7.19}$$

to write the equations (7.17) and (7.18) in a compact form

$$2 \int_0^\pi d\sigma' \Lambda_{AB}(\sigma, \sigma') \psi_B^\mu(\sigma') = \psi_A^\mu(\sigma) \quad ; \quad (A, B = -, +). \quad (7.20)$$

From the above equation Λ can be interpreted as a matrix valued “delta function” which acts on the two component Majorana spinor. Instead of (7.5) we therefore propose the following antibrackets in the fermionic sector

$$\{\psi_A^\mu(\sigma), \psi_B^\nu(\sigma')\} = -2i\eta^{\mu\nu} \Lambda_{AB}(\sigma, \sigma'). \quad (7.21)$$

Making use of eq. (7.19) we write this in its component form

$$\begin{aligned} \{\psi_+^\mu(\sigma), \psi_+^\nu(\sigma')\} &= \{\psi_-^\mu(\sigma), \psi_-^\nu(\sigma')\} = -2i\eta^{\mu\nu} \delta_{(a)P}(\sigma - \sigma') \\ \{\psi_-^\mu(\sigma), \psi_+^\nu(\sigma')\} &= -2i\eta^{\mu\nu} \delta_{(a)P}(\sigma + \sigma'). \end{aligned} \quad (7.22)$$

Remarkably the above set of antibracket algebra is now completely consistent with the BC(s). To see this explicitly, we compute the anticommutator of $\psi_+^\nu(\sigma')$ with (7.10) and (7.11), the left hand side of which gives:

$$\begin{aligned} -2i \left(\delta_{(a)P}(\sigma - \sigma') - \delta_{(a)P}(\sigma + \sigma') \right) |_{\sigma=0} &= -2i \Delta_{-(a)}(\sigma, \sigma') |_{\sigma=0} \\ &= \frac{i}{\pi} \sum_{n \in Z + \frac{1}{2}} \sin(n\sigma) \sin(n\sigma') |_{\sigma=0} = 0 \end{aligned} \quad (7.23)$$

$$\begin{aligned} -2i \left(\delta_{(a)P}(\sigma - \sigma') + \delta_{(a)P}(\sigma + \sigma') \right) |_{\sigma=\pi} &= -2i \Delta_{+(a)}(\sigma, \sigma') |_{\sigma=\pi} \\ &= -\frac{i}{\pi} \sum_{n \in Z + \frac{1}{2}} \cos(n\sigma) \cos(n\sigma') |_{\sigma=\pi} = 0 \end{aligned} \quad (7.24)$$

where the form of the antiperiodic delta function (7.15) has been used. This completes the analysis of the fermionic algebra for the NS BC(s). In the next section we shall use these relations (7.22) to compute the super constraint algebra.

7.1.2 Bosonic sector

Let us now study the bosonic sector of the superstring action (7.1). Varying the bosonic part of the action (7.1), we obtain the equation of motion for the bosonic field

$$(\partial_\sigma^2 - \partial_\tau^2) X^\mu = 0 \quad (7.25)$$

together with Dirichlet and Neumann BC(s)

$$\begin{aligned}\delta X^\mu|_{\sigma=0,\pi} &= 0 \\ X'^\mu|_{\sigma=0,\pi} &= 0.\end{aligned}\tag{7.26}$$

Now there are two cases depending on the periodicity of the bosonic variable X^μ . Usually, one is interested in theories with maximum Poincaré invariance and hence X^μ must be periodic (with a periodicity of 2π). This case has already been discussed in previous chapters. On the other hand antiperiodicity of X^μ is interesting because one encounters it for twisted strings on an orbifold [83]. In this chapter we shall discuss this case in details.

We let the bosonic string coordinates $X^\mu(\sigma)$ to have a periodicity of 4π (antiperiodicity of 2π)²:

$$X^\mu(\sigma + 4\pi) = X^\mu(\sigma).\tag{7.27}$$

Hence the integral (7.16) once again holds for the bosonic coordinate $X^\mu(\sigma)$. Restricting to the case of even(odd) functions $X_\pm^\mu(-\sigma) = \pm X_\pm^\mu(\sigma)$, it can be easily seen that (7.16) reduces to:

$$2 \int_0^\pi d\sigma' \Delta_{\pm(a)}(\sigma, \sigma') X_\pm^\mu(\sigma') = X_\pm^\mu(\sigma)\tag{7.28}$$

where $\Delta_{\pm(a)}$ were defined in the eqs. (7.23) and (7.24). We therefore propose the following equal time PB:

$$\{X^\mu(\tau, \sigma), \Pi_\nu(\tau, \sigma')\} = 2 \delta_\nu^\mu \Delta_{\pm(a)}(\sigma, \sigma').\tag{7.29}$$

It is now easy to observe that for $\Delta_{+(a)}(\sigma, \sigma')$ to appear in the above PB the end points must satisfy following BC(s)

$$\begin{aligned}X'^\mu(0) &= 0 \\ X^\mu(\pi) &= 0\end{aligned}\tag{7.30}$$

and for $\Delta_{-(a)}(\sigma, \sigma')$, the appropriate BC(s) that the end points must satisfy, reads

$$\begin{aligned}X^\mu(0) &= 0 \\ X'^\mu(\pi) &= 0.\end{aligned}\tag{7.31}$$

²Note that this is also in accord with the fermionic sector.

We shall find in the next section that the symplectic structure of the bosonic sector also plays a crucial role in the closure of the super constraint algebra.

7.2 Super constraint algebra

In this section we shall compute the algebra of the super-Virasoro constraints using the modified symplectic structures derived in the first section 2.

The complete set of super constraints are given by [60, 79]:

$$\begin{aligned}\chi_1(\sigma) &= \Phi_1(\sigma) + \lambda_1(\sigma) = 0 \\ \chi_2(\sigma) &= \Phi_2(\sigma) + \lambda_2(\sigma) = 0\end{aligned}\tag{7.32}$$

where,

$$\begin{aligned}\Phi_1(\sigma) &= \left(\Pi^2(\sigma) + (\partial_\sigma X(\sigma))^2 \right) \\ \Phi_2(\sigma) &= (\Pi(\sigma) \partial_\sigma X(\sigma)) \\ \lambda_1(\sigma) &= -i \bar{\psi}^\mu(\sigma) \rho_1 \partial_\sigma \psi_\mu(\sigma) = -i (\psi_-^\mu(\sigma) \partial_\sigma \psi_{\mu-}(\sigma) - \psi_+^\mu(\sigma) \partial_\sigma \psi_{\mu+}(\sigma)) \\ \lambda_2(\sigma) &= -\frac{i}{2} \bar{\psi}^\mu(\sigma) \rho_0 \partial_\sigma \psi_\mu(\sigma) = \frac{i}{2} (\psi_-^\mu(\sigma) \partial_\sigma \psi_{\mu-}(\sigma) + \psi_+^\mu(\sigma) \partial_\sigma \psi_{\mu+}(\sigma))\end{aligned}\tag{7.33}$$

and using the basic algebra of fermionic and bosonic variables (7.22, 7.29), we get the following algebra for super-Virasoro constraints:

$$\begin{aligned}\{\chi_1(\sigma), \chi_1(\sigma')\} &= 8 \left(\chi_2(\sigma) \partial_\sigma \Delta_{+(a)}(\sigma, \sigma') + \chi_2(\sigma') \partial_{\sigma'} \Delta_{-(a)}(\sigma, \sigma') \right) \\ \{\chi_2(\sigma), \chi_2(\sigma')\} &= 2 \left(\chi_2(\sigma') \partial_{\sigma'} \Delta_{+(a)}(\sigma, \sigma') + \chi_2(\sigma) \partial_\sigma \Delta_{-(a)}(\sigma, \sigma') \right) \\ \{\chi_2(\sigma), \chi_1(\sigma')\} &= 2 (\chi_1(\sigma) + \chi_1(\sigma')) \partial_\sigma \Delta_{+(a)}(\sigma, \sigma') .\end{aligned}\tag{7.34}$$

Apart from a numerical factor the above algebra has the same structure as in 5th chapter with the only difference that $\delta_P(\sigma)$ occurring in chapter 5 has been replaced by $\delta_{(a)P}(\sigma)$. Similarly one can show that the algebra of super currents

$$\begin{aligned}\tilde{J}_1(\sigma) &= 2J_{01}(\sigma) = \psi_-^\mu(\sigma) \Pi_\mu(\sigma) - \psi_-^\mu(\sigma) \partial_\sigma X_\mu \\ \tilde{J}_2(\sigma) &= 2J_{02}(\sigma) = \psi_+^\mu(\sigma) \Pi_\mu(\sigma) + \psi_+^\mu(\sigma) \partial_\sigma X_\mu\end{aligned}\tag{7.35}$$

among themselves and also with the super constraints (7.32) close. It is also interesting to note that both $\Delta_{+(a)}$ and $\Delta_{-(a)}$ appearing in the PB of the bosonic variables (7.29) gives the same constraint algebra (7.34). Furthermore, the closure of the algebra also indicates the internal consistency of our analysis.

7.3 Mode expansions and symplectic algebra

In this section, we shall derive the fermionic algebra (7.22) and the bosonic algebra (7.29) from a mode expansion of the constituting fields. To do that we consider the mode expansions of the fermionic field (7.12). Here d_n^μ are Fourier modes and they satisfy the algebra

$$\{d_m^\mu, d_n^\nu\} = -\frac{i}{\pi} \eta^{\mu\nu} \delta_{m+n,0}. \quad (7.36)$$

This algebra can be obtained just by following the procedure of [59], in which they have computed the anti brackets among Fourier components of fermionic sector of superstrings (R sector) using Faddeev-Jackiw symplectic formalism [52]. This relation (7.36) between d 's can also be worked out from the contour argument (discussed in chapter 6) [83] and the operator product expansion. The antibracket relations between $\psi_A^\mu(\sigma), \psi_B^\nu(\sigma')$ are then obtained by using (7.12) and (7.36)

$$\begin{aligned} \{\psi_-^\mu(\sigma), \psi_+^\nu(\sigma')\} &= \frac{1}{2} \sum_{r,s \in Z + \frac{1}{2}} e^{-ir(\tau-\sigma)} e^{-is(\tau+\sigma)} \{d_r^\mu, d_s^\nu\} \\ &= -\frac{i}{2\pi} \eta^{\mu\nu} \sum_{r \in Z + \frac{1}{2}} e^{-ir(\tau-\sigma)} e^{ir(\tau+\sigma)} \\ &= -2i\eta^{\mu\nu} \delta_{(a)P}(\sigma + \sigma'). \end{aligned} \quad (7.37)$$

Proceeding exactly in the similar manner one can get back the other anti-brackets of (7.22).

In order to study the bosonic sector, we first need the expressions of the mode expansion for the two different types of BC(s) (7.30) and (7.31).

For the first case (BC (7.30)) it is given by:

$$X^\mu(\tau, \sigma) = \sum_{n \in Z + \frac{1}{2}} \frac{\alpha_n^\mu}{n} e^{in\tau} \sin n\sigma \quad (7.38)$$

and for the other case (BC (7.31)) the mode expansion is

$$X^\mu(\tau, \sigma) = \sum_{n \in Z + \frac{1}{2}} \frac{\alpha_n^\mu}{n} e^{in\tau} \cos n\sigma. \quad (7.39)$$

The canonical momenta corresponding to (7.38) and (7.39) are given by

$$\begin{aligned} \Pi_\mu(\tau, \sigma) &= \eta_{\mu\nu} \partial_\tau X^\nu(\tau, \sigma) \\ &= i\eta_{\mu\nu} \sum_{n \in Z + \frac{1}{2}} \alpha_n^\nu e^{in\tau} \sin n\sigma, \quad i\eta_{\mu\nu} \sum_{n \in Z + \frac{1}{2}} \alpha_n^\nu e^{in\tau} \cos n\sigma. \end{aligned} \quad (7.40)$$

Here also the algebra between the modes can be computed by following the methodology of [51, 52]:

$$\{\alpha_m^\mu, \alpha_n^\nu\} = -\frac{i}{\pi} \eta^{\mu\nu} m \delta_{m+n, 0}. \quad (7.41)$$

Using (7.41) we obtain the same equal time PB given in (7.29).

7.4 The interacting theory

After finishing the analysis for the free theory, we shall now study the interacting case where a superstring moves in the presence of a constant antisymmetric tensor field $\mathcal{B}_{\mu\nu}$. The action given by [57, 81]:

$$\begin{aligned} S &= \frac{-1}{2} \int_{\Sigma} d\tau d\sigma \left[\partial_a X^\mu \partial^a X_\mu + \epsilon^{ab} \mathcal{B}_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \right. \\ &\quad \left. + i\psi_{\mu-} E^{\nu\mu} \partial_+ \psi_{\nu-} + i\psi_{\mu+} E^{\nu\mu} \partial_- \psi_{\nu+} \right] \end{aligned} \quad (7.42)$$

where, $\partial_+ = \partial_\tau + \partial_\sigma$, $\partial_- = \partial_\tau - \partial_\sigma$ and $E^{\mu\nu} = \eta^{\mu\nu} + \mathcal{B}^{\mu\nu}$. Now since the bosonic and fermionic sectors decouple, we can study them separately.

Here we concentrate on the fermionic sector. The variation of the fermionic part of the action (7.42) gives the classical equations of motion:

$$\partial_+ \psi_{\nu-} = 0 \quad , \quad \partial_- \psi_{\nu+} = 0 \quad (7.43)$$

and a boundary term that yields the following NS BC(s)³:

$$\begin{aligned} E_{\nu\mu} \psi_+^\nu(0, \tau) &= E_{\mu\nu} \psi_-^\nu(0, \tau) \\ E_{\nu\mu} \psi_+^\nu(\pi, \tau) &= -E_{\mu\nu} \psi_-^\nu(\pi, \tau) \end{aligned} \quad (7.44)$$

at the endpoints $\sigma = 0$ and $\sigma = \pi$ of the string.

As in the free case, the above non-trivial BC(s) leads to a modification in the symplectic structure (7.5). The $\{\psi_{(\pm)}^\mu(\sigma, \tau), \psi_{\pm}^\nu(\sigma', \tau)\}$ is the same as (7.22). In the case of mixed bracket, we make the following ansatz:

$$\{\psi_+^\mu(\sigma, \tau), \psi_-^\nu(\sigma', \tau)\} = C^{\mu\nu} \delta_{(a)P}(\sigma + \sigma') . \quad (7.45)$$

Brackets $\psi_-^\gamma(\sigma')$ with the BC(s) (7.44) one obtains

$$E_{\nu\mu} C^{\nu\gamma} = -2i E_{\mu\gamma} \quad (7.46)$$

which on solving gives

$$C^{\mu\nu} = -2i \left[(1 - \mathcal{B}^2)^{-1} \right]^{\mu\rho} E_{\rho\gamma} E^{\gamma\nu} . \quad (7.47)$$

Above solution is written in a matrix notation as,

$$C = -2i \left[(1 - \mathcal{B}^2)^{-1} (1 + \mathcal{B})^2 \right] \quad (7.48)$$

where $C = \{C^{\mu\nu}\}$. Thus we get the modified mixed bracket in the form

$$\{\psi_+^\mu(\sigma, \tau), \psi_-^\nu(\sigma', \tau)\} = -2i \left[(1 - \mathcal{B}^2)^{-1} \right]^{\mu\rho} E_{\rho\gamma} E^{\gamma\nu} \delta_{(a)P}(\sigma + \sigma') . \quad (7.49)$$

If we take the limit $\mathcal{B}_{\mu\nu} \rightarrow 0$ in the above equation we get back the last relation of (7.22).

7.5 Summary

In string theory the modification of Poisson algebra is a consequence of the nontrivial BC(s). In this chapter, we have studied this problem for an open superstring satisfying the NS BC(s).

³The boundary term also leads to R BC(s). Detailed investigations involving R BC(s) has already been carried out in chapter 5 and 6.

Here also we have obtained non(anti)commutative structure for the fermionic string coordinates following the approach discussed in chapter 5. So in that sense this is an extension of the chapter 5 and 6.

Chapter 8

Concluding remarks

Noncommutativity of spacetime and its consequences for quantum field theory have been one of the main objects of interest in the last few years. An important source of noncommutativity in string theory is the presence of an antisymmetric constant tensor field along the D-brane world volumes (where the string end points are located). The quantisation of strings attached to branes involves mixed (combination of Dirichlet and Neumann) boundary conditions. This makes the quantisation procedure more subtle since the quantum commutators must be consistent with these boundary conditions. The aim of this thesis is to go further with these investigations by making a thorough study on the role played by boundary conditions in noncommutativity/non(anti)commutativity in string theory.

We started, in chapter 1, with a brief introduction of how noncommutativity appears in string theory. Different approaches have been adopted to obtain this result. In some of the earlier papers in the literature, the authors have regarded the boundary conditions as constraints. The interpretation of the boundary condition as primary constraints usually lead to an infinite tower of second class constraints in contrast to the usual Dirac formulation of constrained systems. Besides, in this approach, where one tries to obtain non-commutativity through Dirac brackets between coordinates, one encounters ambiguous factor like $\delta(0)$. Furthermore, different results are obtained depending on the interpretations of these factors. On the other hand Hanson, Regge and Teitelboim [41], modified the canonical Poisson bracket structure, so that

it is compatible with the boundary conditions. The modified Poisson brackets were obtained for the free NG string, in the orthonormal gauge, which is the counterpart of the conformal gauge in the free Polyakov string. We essentially followed the same procedure in chapter 2, to modify the basic brackets of Polyakov string so that it is compatible with the boundary conditions.

In chapter 2, we first presented a review of noncommutativity in an open string moving in a background Neveu-Schwarz field in a gauge independent hamiltonian approach. The noncommutativity was seen to be a direct consequence of the nontrivial boundary conditions, which in contrary to several approaches, were not treated as constraints. The origin of any modification in the usual Poisson algebra was the presence of boundary conditions. In a gauge independent formulation of a free Polyakov string, the boundary conditions naturally led to a noncommutative structure among the string coordinates. This noncommutativity vanished in the conformal gauge, as expected. For the interacting string, a more involved boundary condition led to a more general type of noncommutativity. In contrary to the standard conformal gauge expressions, this noncommutative structure survived at all points of the string and not just at the boundaries. Also in contrast to the free string theory, this noncommutativity could not be entirely removed in any gauge. In the conformal gauge, noncommutativity survived only at the string end points.

We then discussed a new form of the action that interpolates between the Nambu-Goto and Polyakov form of interacting bosonic string, without the need of any gauge fixing. The interpolating Lagrangian introduced in this chapter contains as many fields as there are independent degrees of freedom. Being already in the first order form, this action is free from nonlinearity problems associated with the Nambu-Goto action. It also does not contain redundant fields as in the Polyakov forms. Here also it was seen that the basic brackets are not compatible with the interpolating boundary conditions. So we modified the basic Poisson brackets in order to establish consistency of the boundary condition with the basic Poisson brackets. A thorough analysis of the gauge symmetries of interpolating actions was then performed in this noncommutative set up using a general method based on Dirac's theory of constrained Hamiltonian analysis. Specifically we demonstrated the equivalence of the reparametrisation invariances of

different string actions with the gauge invariances generated by the first class constraints. Indeed, the whole analysis of the interpolating Lagrangian formalism was based on the local gauge symmetries only. Finally, we feel that it would be interesting to investigate whether non-critical strings can be discussed using the interpolating action in a path-integral framework.

So far we basically discussed the appearances of noncommutativity in bosonic string at classical level. So we extended our analysis to the quantum level in chapter 4. We first discussed new normal ordered products for open string position operators that satisfy both the equations of motion and the boundary conditions. Using the contour argument and the new X - X operator product expansion we calculated the commutator among the Fourier components and then the commutation relations among string coordinates. In this chapter, we used conformal field theory techniques to compute the commutator among Fourier components which was unlike the method in ([51]), where the algebra among the Fourier components were computed using the Faddeev-Jackiw symplectic formalism. This was then used to obtain the commutator between the basic fields. The advantage of this approach was that the results one obtained took into account the quantum effects right from the beginning, in contrary to the previous investigations, which were made essentially at the classical level, so that the question of the existence of quantum effects, if any, can be addressed immediately. For example, it was checked that the new normal ordering, as proposed in [55], which took into account the boundary conditions had no bearing on the central charge in case of free bosonic string. Finally, we also computed the oscillator algebra in presence of the B field which is a parity-odd field on the string world-sheet. Consequently in presence of this B field, the left and right moving modes appearing in the Laurent series expansions of the (anti)holomorphic fields ∂X^μ and $\bar{\partial} X^\mu$ (4.23) of the closed string were no longer equal when open string BCs were imposed to obtain the corresponding Laurent expansions. These rather got related to the free oscillator modes γ_m^μ (4.34) in a parity asymmetric way. Using these expressions of left and right moving modes, we rewrote the (anti)holomorphic fields ∂X^μ and $\bar{\partial} X^\mu$ entirely in terms of the free oscillator modes γ_m^μ (4.35). Then a straight forward calculation, involving XX OPE and contour argument yield the NC commutator given in (4.43), thereby reproducing the previous chapters results and that of [7, 8, 10, 39, 51], even though we had made use of newly proposed normal ordering [55] which

was compatible with boundary conditions.

We then extended our analysis from bosonic to the superstring case in the next few chapters. As pointed out earlier the origin of any modification in the usual canonical algebra is the presence of boundary conditions. This phenomenon is quite well known for a free scalar field subjected to periodic boundary conditions. Besides this method was also used earlier by [41] in the context of Nambu-Goto formulation of the bosonic string. We show that the same thing also held true in the fermionic sector of the conformal gauge fixed free superstring. It should be mentioned that in the case of fermionic string there is a choice between two boundary conditions *viz* Ramond boundary conditions and Neveu Schwarz boundary conditions. In chapter 5 and 6 we worked in detail with Ramond boundary conditions and finally in chapter 7 we worked with Neveu Schwarz boundary conditions. Here also the boundary conditions became periodic once we extended the domain of definition of the length of the string from $[0, \pi]$ to $[-\pi, \pi]$. This mathematical trick led to a modification where the usual Dirac delta function got replaced by a periodic delta function. Eventually one constructs the appropriate “delta function” for the physical interval $[0, \pi]$ of the string to write down the basic symplectic structure. Interestingly, there we got a 2×2 matrix valued “delta function” appropriate for the two component Majorana spinor. This is in contrast to the bosonic case, where one had a single component “delta function” $\Delta_+(\sigma, \sigma')$ satisfying Neumann boundary condition. This symplectic structure, interestingly, led to a new involutive structure for the super-Virasoro algebra at the classical level. The interesting thing to be noted is that, unlike the bosonic case, we got an anticommutative structure in the fermionic sector even for the free superstring. Our results differ from those in [58] and were mathematically consistent which was reflected from the closure of the constraint algebras. The analysis of this chapter is a direct generalisation of bosonic string discussed in 2nd and 3rd chapter. The same technique was adopted for the interacting case also where the boundary condition got more involved and led to a more general type of non(anti)-commutativity that had been observed before. However, our results were once again different from the existing results since we got a periodic delta function instead of the usual delta function, apart from the relative sign of σ, σ' . This change of relative sign indeed played a crucial role in the internal consistency of our analysis. Further, the interacting results

go over smoothly to the free case once the interaction was switched off.

It is important to note that all these discussion in the context of superstrings were at classical level. So in the next chapter we calculated the normal ordered products for fermionic open string coordinates in the presence of an antisymmetric tensor background taking the boundary conditions into account. Then we again computed the anticommutator between the basic fermionic fields using the conformal field theoretic techniques. In that sense this was an extension of chapter 4 (in which we discussed normal ordering for bosonic string coordinates only).

Finally we extended our methodology to the superstring satisfying the Neveu Schwarz boundary conditions in chapter 7. Following the approach of 5th chapter, here also the domain of the string length was extended from $[0, \pi]$ to $[-\pi, \pi]$ to get the antiperiodic boundary conditions. That construction enables us to get the 2×2 matrix valued δ function in the algebra of the fermionic sector. Apart from a numerical factor the fermionic algebra was identical to the result obtained in chapter 5. However for the bosonic part of the superstring the result was drastically different. We stress that the symplectic algebra of the bosonic variables, in that chapter contained both $\Delta_{+(a)}(\sigma, \sigma')$ and $\Delta_{-(a)}(\sigma, \sigma')$ (certain combination of anti periodic delta function) which was completely different from the Ramond case where only $\Delta_+(\sigma, \sigma')$ was present. Interestingly that the symplectic structure containing both $\Delta_{\pm(a)}(\sigma, \sigma')$, kept the superconstraint algebra closed provided one imposes Neumann boundary conditions at one end and Dirichlet boundary conditions at the other end of the string in the bosonic sector. That observation was completely new and had not been noticed before in the literature. Finally to complete the analysis, we calculated the non(anti)commutative structures for the interacting case by employing the same procedure. As one expected, without the background field term, the interacting results took the limiting value of the free case

Appendix A

Reality condition for a Majorana spinor

We have chosen a convenient basis of Dirac matrices as

$$\rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (\text{A.1})$$

satisfying the Clifford algebra (5.4). In this representation the component of Ψ_D is given by $\Psi_{D\pm}$

$$\Psi_D^\mu = \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix}. \quad (\text{A.2})$$

Now suppose it satisfies the Dirac equation

$$i\rho^\mu \partial_\mu \Psi_D = 0. \quad (\text{A.3})$$

In the presence of a background electromagnetic field, the corresponding equation is obtained by replacing

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu - ieA_\mu$$

and (A.3) reduces to

$$i\rho^\mu (\partial_\mu - ieA_\mu) \Psi_D = 0. \quad (\text{A.4})$$

Hole theory interpretation ensures that there exists a corresponding solution Ψ_D^c for the antiparticle of charge $(-e)$ satisfying

$$-i\rho^\mu (\partial_\mu + ieA_\mu) \Psi_D^c = 0. \quad (\text{A.5})$$

Defining $\bar{\Psi} = \Psi^\dagger \rho_0$ we find the conjugate Dirac equation as

$$-i\rho^{\mu T}(\partial_\mu + ieA_\mu)\bar{\Psi}_D^T = 0. \quad (\text{A.6})$$

Now in order that the matrices $-\rho^{\mu T}$ also satisfy (the Clifford algebra) (5.4), and there must exist a nonsingular matrix C such that

$$C^{-1}\rho^\mu C = -\rho^{\mu T} \quad (\text{A.7})$$

so that (A.4) matches with (A.6). Thus, if we define the ‘charge-conjugate spinor’ Ψ_D^c by putting

$$\Psi_D^c = C\bar{\Psi}_D^T \quad (\text{upto a phase}), \quad (\text{A.8})$$

we see that it satisfies (A.5). It is easy to show that C is always antisymmetric and, in this representation (A.1), we may choose C to be (proportional to) $\sigma^3\rho^1$, i.e.¹,

$$C = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \quad (\text{A.9})$$

A Majorana spinor Ψ_M is defined as one that equals its charge-conjugate spinor, i.e., $\Psi_M^c = \Psi_M$. For a Majorana spinor we therefore have

$$\begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix} = \begin{pmatrix} \psi_-^{\mu *} \\ \psi_+^{\mu *} \end{pmatrix}, \quad (\text{A.10})$$

which is the reality condition.

Now we proceed to establish a relation between the chiral representation and the representation in this chapter, of a Majorana spinor. In chiral representation

$$\{\Psi_M\}_c = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \quad (\text{A.11})$$

and the γ matrices satisfying the Clifford algebra (5.4) in the chiral representation read

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (\text{A.12})$$

¹where $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Let S be a matrix such that

$$\begin{aligned}\rho^a &= S\gamma^a S^{-1}, \\ \Psi &= S\{\Psi_M\}_c,\end{aligned}\tag{A.13}$$

where Ψ is given by (5.9). This immediately leads to the following solution for the matrix S :

$$S = \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}.\tag{A.14}$$

Hence, from (A.13), we have

$$\psi_L(\sigma) = \psi_-(\sigma), \quad -i\psi_R(\sigma) = \psi_+(\sigma).\tag{A.15}$$

Clearly it follows that $\psi_L(\sigma)$ is real but $\psi_R(\sigma)$ is purely imaginary. Also one can easily identify $\psi_+(\sigma)$ and $\psi_-(\sigma)$ to be the real chiral components themselves. Therefore from physical grounds one can easily expect

$$\psi_+(-\sigma) = \psi_-(\sigma).\tag{A.16}$$

Appendix B

Computational details of some of the key result of the chapter 6

Here we would like to give some of the computational details involved in deriving (6.13) from (6.7) and (6.12) (for convenience we treat the free case, i.e. $\mathcal{B} = 0$). Eq.(6.7) with z replaced by ω yields:

$$\begin{aligned} 0 &= \int [d\psi] \left[\frac{\delta}{\delta\psi_{(a)}^\mu(\omega, \bar{\omega})} [e^{-S_F} \psi_{(b)}^\nu(\omega', \bar{\omega}')] \right] \\ &= \int [d\psi] e^{-S_F} \left[-\frac{\delta S_F}{\delta\psi_{(a)}(\omega, \bar{\omega})} \psi_{(b)}(\omega', \bar{\omega}') + \frac{\delta\psi_{(b)}(\omega', \bar{\omega}')}{\delta\psi_{(a)}(\omega, \bar{\omega})} \right] \end{aligned} \quad (\text{B.1})$$

Putting $a = +$, $b = -$; we obtain:

$$\begin{aligned} 0 &= \int [d\psi] e^{-S_F} \left[\frac{i}{2\pi\alpha'} \partial_\omega \psi_+(\omega, \bar{\omega}) \psi_-(\omega', \bar{\omega}') + \frac{\delta\psi_{(-)}(\omega', \bar{\omega}')}{\delta\psi_{(+)}(\omega, \bar{\omega})} \right. \\ &\quad \left. + \frac{i}{4\pi\alpha'} \oint_{\partial\Sigma} d\omega'' \delta^2(\omega'' - \omega, \bar{\omega}'' - \bar{\omega}) \psi_{(-)}(\omega', \bar{\omega}') \right. \\ &\quad \left. \times \left(\psi_{(-)}(\omega'', \bar{\omega}'') - i\psi_{(+)}(\omega'', \bar{\omega}'') \right) \right] \end{aligned} \quad (\text{B.2})$$

Now we discuss two distinct cases separately.

- Case 1: The insertion $\psi_{(-)}(\omega', \bar{\omega}')$ is not located at the boundary:

In this case

$$\frac{\delta\psi_{(-)}(\omega', \bar{\omega}')}{\delta\psi_{(+)}(\omega, \bar{\omega})} = 0 \quad (\text{B.3})$$

and therefore one finds:

$$\langle \partial_\omega \psi_{(+)}^\mu(\omega, \bar{\omega}) \psi_{(-)}^\nu(\omega', \bar{\omega}') \rangle = 0. \quad (\text{B.4})$$

• Case 2: The insertion $\psi_{(-)}(\omega')$ is located at the boundary (since $\omega' = \bar{\omega}'$ at the boundary, the insertion $\psi_{(-)}(\omega')$ depends only on the argument ω'):

In this case the computation of the second term in (B.2) needs to be done more carefully. One finds

$$\begin{aligned} \frac{\delta \psi_{(-)}(\omega', \bar{\omega}')}{\delta \psi_{(+)}(\omega, \bar{\omega})} \Big|_{\omega'=\bar{\omega}'} &= i \frac{\delta \psi_{(+)}(\omega', \bar{\omega}')}{\delta \psi_{(+)}(\omega, \bar{\omega})} \Big|_{\omega'=\bar{\omega}'} \\ &= i \delta^2 (\omega - \omega', \bar{\omega} - \bar{\omega}') \Big|_{\omega'=\bar{\omega}'} \\ &= i \delta^2 (\omega - \omega', \bar{\omega} - \omega') . \end{aligned} \quad (\text{B.5})$$

where we have used the BC (6.12) (with ω repaced by ω') in the first line of (B.5).

Substituting (B.5) in (B.2) and equating the volume term to zero, one finds the third of the equations in (6.13) (with $\mathcal{B} = 0$).

Similarly, for other choices of a, b the rest of the equations in (6.13) can be derived (with $\mathcal{B} = 0$).

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